Linear Algebra and its Applications 624 (2021) 44–52

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

The extension dimension of triangular matrix algebras

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A R T I C L E   I N F O

Article history:
Received 26 September 2020
Accepted 2 April 2021
Available online xxxx
Submitted by M. Bresar

MSC:
18G20
16E10
18E10

Keywords:
Extension dimension
Weak resolution dimension
Triangular matrix algebras

A B S T R A C T

Let T, U be two Artin algebras and ${}_UM_T$ be a U-T-bimodule. In this paper, we study the extension dimension of the formal triangular matrix algebra $\Lambda = \begin{pmatrix} T & 0 \\ M & U \end{pmatrix}$. It is proved that if ${}_UM$, M_T are projective and $\max\{\operatorname{gl.dim} T, \dim \operatorname{mod} U\} \geqslant 1$, then

$$\max\{\dim \operatorname{mod} T, \dim \operatorname{mod} U\}$$
$$\leqslant \dim \operatorname{mod} \Lambda \leqslant \max\{\operatorname{gl.dim} T, \dim \operatorname{mod} U\}.$$

1. Introduction

Rouquier introduced in [16,17] the dimension of a triangulated category under the idea of Bondal and van den Bergh in [6]. Roughly speaking, it is an invariant that measures how quickly the category can be built from one object. This dimension plays

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https://doi.org/10.1016/j.laa.2021.04.002
0024-3795/

an important role in representation theory ([3,5,8,11,15–17,19]). For example, it can be used to compute the representation dimension of Artin algebras ([16,14]). Similar to the dimension of triangulated categories, the (extension) dimension of an abelian category was introduced by Beligiannis in [4], also see [9]. The size of the extension dimension reflects how far an Artin algebra is from a finite representation type, some relate result can be see [20] and so on.

Given an Artin algebra Λ, we denote the category of finitely generated left Λ-modules by $\operatorname{mod}\Lambda$, the dimension of $\operatorname{mod}\Lambda$ by $\dim\operatorname{mod}\Lambda$. $LL(\Lambda)$ will be the Loewy length of Λ, $\operatorname{gl.dim}\Lambda$ the global dimension of Λ, and $\operatorname{w.resol.dim}\operatorname{mod}\Lambda$ the weak resolution dimension of Λ (see [14]). Moreover, $\operatorname{rep.dim}\Lambda$ will be the representation dimension of Λ for an non-semisimple Artin algebra Λ (see [2]). These dimensions have the following relation

$$\operatorname{w.resol.dim}\operatorname{mod}\Lambda = \dim\operatorname{mod}\Lambda \leqslant \max\{LL(\Lambda)-1, \operatorname{gl.dim}\Lambda, \operatorname{rep.dim}\Lambda - 2\}.$$

Also, the extension dimension and finitistic dimension of Λ have the following relation (see [20, Corollary 3.12])

$$\text{if } \dim\operatorname{mod}\Lambda \leqslant 1, \text{then the finitistic dimension of } \Lambda \text{ is finite.}$$

In [18], Wei introduced the notion of n-Igusa-Todorov algebra. The relation between extension dimension and 0-Igusa-Todorov algebra is that Artin algebra Λ is 0-Igusa-Todorov algebra if and only if $\dim\operatorname{mod}\Lambda \leqslant 1$ (see [20, Proposition 3.15]).

Given two Artin algebra Γ and Λ, if Γ is an excellent extension (see [12]), then $\dim\operatorname{mod}\Lambda = \dim\operatorname{mod}\Gamma$ (see [20, Theorem 4.2(1)]); if Λ and Γ are separably equivalent (see [13]), then $\dim\operatorname{mod}\Lambda = \dim\operatorname{mod}\Gamma$ (see [20, Theorem 4.5]). These facts perhaps reflect that the extension dimension of Artin algebra may be an good invariant.

In this paper, we will study the dimension of some triangular matrix algebras, the idea comes from [7]. Our main result is as follows.

Theorem 1.1. *Let* $\Lambda = \begin{pmatrix} T & 0 \\ {}_UM_T & U \end{pmatrix}$ *be a triangular matrix Artin R-algebra such that* ${}_UM \in \operatorname{add}(U)$, $M_T \in \operatorname{add}(T)$ *and* $\max\{\operatorname{gl.dim}T, \dim\operatorname{mod}U\} \geqslant 1$. *Then*

$$\max\{\dim\operatorname{mod}T, \dim\operatorname{mod}U\} \leqslant \dim\operatorname{mod}\Lambda \leqslant \max\{\operatorname{gl.dim}T, \dim\operatorname{mod}U\}.$$

2. Preliminaries

2.1. The extension dimension of abelian category

Let $\mathcal{A}$ be an abelian category. All subcategories of $\mathcal{A}$ are full, additive and closed under isomorphisms and all functors between categories are additive. For a subclass $\mathcal{U}$ of $\mathcal{A}$, we use $\operatorname{add}\mathcal{U}$ to denote the subcategory of $\mathcal{A}$ consisting of direct summands of finite direct sums of objects in $\mathcal{U}$.

Let $\mathcal{U}_1, \mathcal{U}_2, \cdots, \mathcal{U}_n$ be subcategories of $\mathcal{A}$. Define

$$\mathcal{U}_1 \bullet \mathcal{U}_2 := \operatorname{add}\{A \in \mathcal{A} \mid \text{there exists an sequence } 0 \to U_1 \to A \to U_2 \to 0 \\ \text{in } \mathcal{A} \text{ with } U_1 \in \mathcal{U}_1 \text{ and } U_2 \in \mathcal{U}_2\}.$$

Inductively, define

$$\mathcal{U}_1 \bullet \mathcal{U}_2 \bullet \cdots \bullet \mathcal{U}_n := \operatorname{add}\{A \in \mathcal{A} \mid \text{there exists an sequence } 0 \to U \to A \to V \to 0 \\ \text{in } \mathcal{A} \text{ with } U \in \mathcal{U}_1 \text{ and } V \in \mathcal{U}_2 \bullet \cdots \bullet \mathcal{U}_n\}.$$

For a subcategory $\mathcal{U}$ of $\mathcal{A}$, set $[\mathcal{U}]_0 = 0$, $[\mathcal{U}]_1 = \operatorname{add}\mathcal{U}$, $[\mathcal{U}]_n = [\mathcal{U}]_1 \bullet [\mathcal{U}]_{n-1}$ for any $n \geqslant 2$, and $[\mathcal{U}]_\infty = \bigcup_{n \geqslant 0} [\mathcal{U}]_n$ ([4]). For any subcategories $\mathcal{U}, \mathcal{V}$ and $\mathcal{W}$ of $\mathcal{A}$, by [9, Proposition 2.2] we have

$$(\mathcal{U} \bullet \mathcal{V}) \bullet \mathcal{W} = \mathcal{U} \bullet (\mathcal{V} \bullet \mathcal{W}).$$

Definition 2.1 *([4]).* The **extension dimension** $\dim \mathcal{A}$ of $\mathcal{A}$ is defined to be

$$\dim \mathcal{A} := \inf\{n \geqslant 0 \mid \mathcal{A} = [T]_{n+1} \text{ with } T \in \mathcal{A}\}.$$

Definition 2.2 *([4]).* Let $M \in \mathcal{A}$. The weak M-resolution dimension of an object X in $\mathcal{A}$, denoted by M-w.resol.dim X, is defined as the minimal positive integer i such that there exists an exact sequence

$$0 \to M_i \to M_{i-1} \to \cdots \to M_0 \to X \to 0$$

in $\mathcal{A}$ with all $M_j \in \operatorname{add} M, 0 \leqslant j \leqslant i$. The weak M-resolution dimension of $\mathcal{A}$, M-w.resol.dim $\mathcal{A}$, is defined as $\sup\{M\text{-w.resol.dim}\, X \mid X \in \mathcal{A}\}$. The weak resolution dimension of $\mathcal{A}$ is denoted by w.resol.dim $\mathcal{A}$ and defined as $\inf\{M\text{-w.resol.dim}\, \mathcal{A} | M \in \mathcal{A}\}$.

Lemma 2.3 *([20]). Assume that $\mathcal{A}$ admits an additive generating object. If $\mathcal{A}$ has enough projective objects and enough injective objects, then*

$$\text{w.resol.dim}\, \mathcal{A} = \dim \mathcal{A}.$$

3. Main result

Let T, U be R-Artin algebras and ${}_U M_T$ a (U, T)-bimodule. Then the triangular matrix algebra

$$\Lambda := \begin{pmatrix} T & 0 \\ M & U \end{pmatrix}$$

can be defined by the ordinary operation on matrices. Let $\mathcal{C}_\Lambda$ be the category whose objects are the triples (A, B, f) with $A \in \operatorname{mod} T$, $B \in \operatorname{mod} U$ and $f \in \operatorname{Hom}_U(M \otimes_T A, B)$. The morphisms from (A, B, f) to (A', B', f') are pairs of (α, β) such that the following diagram

$$\begin{tikzcd} M \otimes_T A \arrow[r, "f"] \arrow[d, "M\otimes\alpha"'] & B \arrow[d, "\beta"] \\ M \otimes_T A' \arrow[r, "f'"] & B' \end{tikzcd}$$

commutes, where $\alpha \in \operatorname{Hom}_T(A, A')$ and $\beta \in \operatorname{Hom}_U(B, B')$.

It is well known that there exists an equivalence of categories between $\operatorname{mod} \Lambda$ and $\mathcal{C}_\Lambda$ ([10]). Hence we can view a Λ-module as a triples (A, B, f) with $A \in \operatorname{mod} T$ and $B \in \operatorname{mod} U$. Moreover, a sequence

$$0 \to (A_1, B_1, f_1) \xrightarrow{(\alpha_1,\beta_1)} (A_2, B_2, f_2) \xrightarrow{(\alpha_2,\beta_2)} (A_3, B_3, f_3) \to 0$$

in $\operatorname{mod} \Lambda$ is exact if and only if

$$0 \to A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} A_3 \to 0$$

is exact in $\operatorname{mod} T$ and

$$0 \to B_1 \xrightarrow{\beta_1} B_2 \xrightarrow{\beta_2} B_3 \to 0$$

is exact in $\operatorname{mod} U$. All indecomposable projective modules in $\operatorname{mod} \Lambda$ are exactly of the forms $(P, M \otimes P, 1)$ and $(0, Q, 0)$, where P is an indecomposable projective T-module and Q is an indecomposable projective U-module.

Lemma 3.1. *Let $\Lambda = \begin{pmatrix} T & 0 \\ {}_U M_T & U \end{pmatrix}$ be a triangular matrix Artin R-algebra such that ${}_U M \in$ $\operatorname{proj}(U)$ and $M_T \in \operatorname{proj}(T)$. Given the left Λ-module (A, B, f) and the following two exact sequences*

$$0 \longrightarrow \Omega_T(A) \xrightarrow{i^A} P_0 \xrightarrow{\pi^A} A \longrightarrow 0$$

$$0 \longrightarrow K_1 \longrightarrow X_0 \xrightarrow{h} B \longrightarrow 0$$

with P_0 the projective cover of A, we have the following exact sequence

$$0 \longrightarrow (\Omega_T(A), K_1 \oplus (M \otimes_T P_0), f_1) \longrightarrow \left(P_0, X_0 \oplus (M \otimes_T P_0), \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)\right)$$
$$\longrightarrow (A, B, f) \longrightarrow 0.$$

Proof. Let $(A, B, f) \in \operatorname{mod}\Lambda$. Since $M_T \in \operatorname{proj}(T)$, the exact sequence

$$0 \longrightarrow \Omega_T(A) \xrightarrow{i^A} P_0 \xrightarrow{\pi^A} A \longrightarrow 0$$

induces an exact sequence

$$0 \longrightarrow M \otimes_T \Omega_T(A) \xrightarrow{M\otimes i^A} M \otimes_T P_0 \xrightarrow{M\otimes \pi^A} M \otimes_T A \longrightarrow 0.$$

Hence, we have the following commutative diagram,

$$\begin{tikzcd}
0 \arrow[r] & M \otimes_T \Omega_T(A) \arrow[r, "M\otimes i^A"] \arrow[d, "f_1"] & M \otimes_T P_0 \arrow[r, "M\otimes \pi^A"] \arrow[d, "\binom{0}{1}"] & M \otimes_T A \arrow[r] \arrow[d, "f"] & 0 \\
0 \arrow[r] & X \arrow[r, "\binom{\alpha}{\beta}"'] & X_0 \oplus (M \otimes_T P_0) \arrow[r, "\left(h, f\circ(M\otimes\pi^A)\right)"'] & B \arrow[r] & 0
\end{tikzcd}$$

where X is a U-module such that the second row is exact. Therefore, we have the pullback diagram

$$\begin{tikzcd}
0 \arrow[r] & K_1 \arrow[r] \arrow[d, equal] & X \arrow[r, "\beta"] \arrow[d, "-\alpha"] & M \otimes_T P_0 \arrow[r] \arrow[d, "f\circ(M\otimes\pi^A)"] & 0 \\
0 \arrow[r] & K_1 \arrow[r] & X_0 \arrow[r, "h"'] & B \arrow[r] & 0
\end{tikzcd}$$

Note that ${}_U M \in \operatorname{proj}(U)$, we have $M \otimes_T P_0$ is projective, and hence the first row in the above diagram splits. Thus $X \cong K_1 \oplus (M \otimes P_0)$.

Hence, we have the following commutative diagram,

$$\begin{tikzcd}
0 \arrow[r] & M \otimes_T \Omega_T(A) \arrow[r, "M\otimes i^A"] \arrow[d, "f_1"] & M \otimes_T P_0 \arrow[r, "M\otimes \pi^A"] \arrow[d, "\binom{0}{1}"] & M \otimes_T A \arrow[r] \arrow[d, "f"] & 0 \\
0 \arrow[r] & K_1 \oplus (M \otimes_T P_0) \arrow[r, "\binom{\alpha}{\beta}"'] & X_0 \oplus (M \otimes_T P_0) \arrow[r, "\left(h, f\circ(M\otimes\pi^A)\right)"'] & B \arrow[r] & 0
\end{tikzcd}$$

which implies the sequence

$$0 \longrightarrow (\Omega_T(A), K_1 \oplus (M \otimes_T P_0), f_1) \longrightarrow \left(P_0, X_0 \oplus (M \otimes_T P_0), \binom{0}{1}\right) \longrightarrow (A, B, f) \longrightarrow 0$$

is exact. $\square$

Now, we can prove our main theorem.

Theorem 3.2. *Let* $\Lambda = \left(\begin{smallmatrix} T & 0 \\ {}_UM_T & U \end{smallmatrix}\right)$ *be a triangular matrix Artin* R*-algebra such that* ${}_UM \in \operatorname{add}(U)$*,* $M_T \in \operatorname{add}(T)$ *and* $\max\{\operatorname{gl.dim} T, \dim\operatorname{mod} U\} \geqslant 1$*. Then*

$$\max\{\dim\operatorname{mod} T, \dim\operatorname{mod} U\} \leqslant \dim\operatorname{mod}\Lambda \leqslant \max\{\operatorname{gl.dim} T, \dim\operatorname{mod} U\}.$$

Proof. We can assume $\operatorname{gl.dim} T = n < \infty$ and $\dim\operatorname{mod} U = m$. By Lemma 2.3, there is a U-module V such that for any $B \in \operatorname{mod} U$, we have the following exact sequence

$$0 \to V_m \to V_{m-1} \to \cdots \to V_0 \to B \to 0$$

with $V_i \in \operatorname{add} V$, $i = 0, 1, 2, \cdots, m$.

Take $l = \max\{\operatorname{gl.dim} T, \dim\operatorname{mod} U\} \geqslant 1$. Given a module $(A, B, f) \in \operatorname{mod}\Lambda$, we have the following two exact sequences

$$0 \to P_l^A \to P_{l-1}^A \to \cdots \to P_0^A \to A \to 0$$

and

$$0 \to V_l \to V_{l-1} \to \cdots \to V_0 \to B \to 0$$

with $P_i^A \in \operatorname{add} T$ and $V_i \in \operatorname{add} V$, $i = 0, 1, 2, \cdots, l$.

Now, set $K_i := \operatorname{Ker}(V_i \to V_{i-1}), 1 \leqslant i \leqslant l-1$, $K_0 := B$ and $K_l := V_l$.

Consider the exact sequences,

$$0 \longrightarrow \Omega_T(A) \longrightarrow P_0^A \longrightarrow A \longrightarrow 0$$

and

$$0 \longrightarrow K_1 \longrightarrow V_0 \longrightarrow K_0 \longrightarrow 0,$$

we have the following exact sequence

$$\begin{aligned} 0 &\longrightarrow \big(\Omega_T(A), K_1 \oplus (M \otimes_T P_0^A), f_1\big) \longrightarrow \big(P_0^A, V_0 \oplus (M \otimes P_0^A), \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)\big) \\ &\longrightarrow (A, B, f) \longrightarrow 0 \end{aligned}$$

by Lemma 3.1. Using Lemma 3.1 to exact sequences

$$0 \longrightarrow \Omega_T^{i+1}(A) \longrightarrow P_i^A \longrightarrow \Omega^i A \longrightarrow 0$$

and

$$0 \longrightarrow K_{i+1} \longrightarrow V_i \oplus (M \otimes_T P_{i-1}^A) \longrightarrow K_i \oplus (M \otimes_T P_{i-1}^A) \longrightarrow 0,$$

we have the following exact sequence

$$
\begin{aligned}
0 &\longrightarrow \big(\Omega_T^{i+1}(A), K_{i+1} \oplus (M \otimes_T P_i^A), f_{i+1}\big) \\
&\longrightarrow \big(P_i^A, V_i \oplus (M \otimes_T P_{i-1}^A) \oplus (M \otimes_T P_i^A), \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)\big) \\
&\longrightarrow (\Omega^i A, K_i \oplus (M \otimes_T P_{i-1}^A), f_i) \longrightarrow 0.
\end{aligned}
$$

Thus, we have a long exact sequence

$$
\begin{aligned}
0 \longrightarrow \big(P_l^A, V_l \oplus (M \otimes_T P_l^A), f_l\big) &\longrightarrow \big(P_{l-1}^A, V_l \oplus (M \otimes_T P_{l-2}^A) \oplus (M \otimes_T P_{l-1}^A), \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)\big) \\
&\longrightarrow \big(P_{l-2}^A, V_l \oplus (M \otimes_T P_{l-3}^A) \oplus (M \otimes_T P_{l-2}^A), \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)\big) \longrightarrow \cdots \\
&\longrightarrow \big(P_0^A, V_0 \oplus (M \otimes P_0^A), \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)\big) \longrightarrow (A, B, f) \longrightarrow 0.
\end{aligned}
$$

Define $W := \bigoplus_{i=1}^{r} (T, V \oplus M, h_i)$, where $h_1, h_2, \cdots, h_r$ are R-generators of $\mathrm{Hom}_U(M, V \oplus M)$. It follows that $\dim \operatorname{mod} \Lambda \leqslant l$.

Note that, $(A, 0, 0) \in \operatorname{mod} \Lambda$, for any $A \in \operatorname{mod} T$. Now, suppose $\dim \operatorname{mod} \Lambda = m$. Hence, there is a module $(X, Y, f) \in \operatorname{mod} \Lambda$ such that for any $A \in \operatorname{mod} T$, there is an exact sequence

$$0 \to (X_m, 0, 0) \to (X_{m-1}, 0, 0) \to \cdots \to (X_0, 0, 0) \to (A, 0, 0) \to 0$$

with $(X_i, 0, 0) \in \operatorname{add}(X, Y, f)$. In particular, the following sequence

$$0 \to X_m \to X_{m-1} \to \cdots \to X_0 \to A \to 0$$

is exact with $X_i \in \operatorname{add} X$. Thus $\dim \operatorname{mod} T \leqslant \dim \operatorname{mod} \Lambda$. Similarly, $\dim \operatorname{mod} U \leqslant \dim \operatorname{mod} \Lambda$. $\square$

Remark 3.3. Let $\Lambda = \begin{pmatrix} T & 0 \\ {}_U M_T & U \end{pmatrix}$ be a triangular matrix Artin R-algebra such that ${}_U M \in \operatorname{add}(U)$, $M_T \in \operatorname{add}(T)$. If $\max\{\operatorname{gl.dim} T, \dim \operatorname{mod} U\} = 0$, then Lemma 3.1 implies $\dim \operatorname{mod} \Lambda \leqslant 1$.

In the following example we show that the bound given in Remark 3.3 is sharp.

Example 3.4. Let Q be the Kronecker quiver: $\bullet \rightrightarrows \bullet$ and $\Lambda = kQ$ the Kronecker algebra. Then Λ can be viewed as the triangular matrix $\begin{pmatrix} T & 0 \\ {}_U M_T & U \end{pmatrix}$, where $T \cong k$, $U \cong k$ and $M \cong k^2$. Hence $\operatorname{gl.dim} \dim T = 0$, $\dim \operatorname{mod} U = 0$. Note that Λ is representation-infinite and $\operatorname{gl.dim} \dim \Lambda = 1$, we have $\dim \operatorname{mod} \Lambda = 1$.

Putting $U = M = T$ in Theorem 3.2, we have the following Corollary.

Corollary 3.5. *Let T be an non-semisimple Artin algebra and $\Lambda = \left(\begin{smallmatrix} T & 0 \\ T & T \end{smallmatrix}\right)$. Then*

$$\dim \operatorname{mod} T \leqslant \dim \operatorname{mod} \Lambda \leqslant \operatorname{gl.dim} T.$$

In particular, if $\dim\operatorname{mod} T = \mathrm{gl.dim}\, T$*, then* $\dim\operatorname{mod}\Lambda = \dim\operatorname{mod} T$*.*

Proof. It follows from $\dim\operatorname{mod} T \leqslant \mathrm{gl.dim}\, T$ for any Artin algebra T ([20, Corollary 3.20]). $\square$

Corollary 3.6. *Let* $\Lambda = \begin{pmatrix} T & 0 \\ {}_UM_T & U \end{pmatrix}$ *be a triangular matrix Artin* R*-algebra such that* ${}_UM \in \mathrm{add}(U)$ *and* $M_T \in \mathrm{add}(T)$*. If* T *is semisimple and* $\dim\operatorname{mod} U \geqslant 1$*, then*

$$\dim\operatorname{mod}\Lambda = \dim\operatorname{mod} U.$$

Corollary 3.7. *Let* $\Lambda = \begin{pmatrix} T & 0 \\ {}_UM_T & U \end{pmatrix}$ *be a triangular matrix Artin* R*-algebra such that* ${}_UM \in \mathrm{add}(U)$ *and* $M_T \in \mathrm{add}(T)$*. If* $\mathrm{gl.dim}\, T \geqslant 1$*, then*

$$\dim\operatorname{mod}\Lambda \leqslant \mathrm{gl.dim}\, T \quad \textit{if and only if} \quad \dim\operatorname{mod} U \leqslant \mathrm{gl.dim}\, T.$$

Let k be a filed. Given a finite dimensional k-algebra U and a U-module ${}_UM$. Recall that the special matrix algebra $\Lambda = \begin{pmatrix} k & 0 \\ M & U \end{pmatrix}$ is said to be the one-point extension of U by M.

Corollary 3.8. *Let* U *be a finite dimensional* k*-algebra and* Λ *be the one-point extension of* U *by a projective* U*-module* M*. If* $\dim\operatorname{mod} U \geqslant 1$*, then*

$$\dim\operatorname{mod}\Lambda = \dim\operatorname{mod} U.$$

In particular, $\dim\operatorname{mod}\Lambda = 1$ *if and only if* $\dim\operatorname{mod} U = 1$*.*

Example 3.9. Let Λ be the k-algebra given by the following quiver

$$\begin{tikzcd}
 & 11 \arrow[d] & & 4 \arrow[dl, "\gamma"'] & 5 \arrow[l, "\epsilon"'] & & & 8 \arrow[dl] \\
1 & 2 \arrow[l, "\alpha"'] & 3 \arrow[l, bend right, "\beta_1"'] \arrow[l, bend left, "\beta_2"] & & & 6 \arrow[lll, "\delta"] \arrow[ul] & 7 \arrow[l, "\lambda"] & 9 \arrow[l] \\
 & & & & & & & 10 \arrow[ul]
\end{tikzcd}$$

with the relations $\beta_i\alpha = \gamma\beta_i = \delta\beta_i = \epsilon\gamma = \lambda\delta = 0$, for $i = 1, 2$.

Let $U := \Lambda/\langle e_{11}\rangle$. By [1, Example 2.8], we know that $\mathrm{rep.dim}\, U = 3$. And by [20, Corollary 3.8(2)], we get $\dim\operatorname{mod} U = 1$. Note that Λ is the one-point extension of U by indecomposable projective U-module P_2 corresponding to the point 2. Now, by Corollary 3.8, we have $\dim\operatorname{mod}\Lambda = 1$.

Declaration of competing interest

There is no competing interest.

Acknowledgements

The authors would like to thank Xin Ma and Yingying Zhang for helpful discussions. They also thank the referee for the useful and detailed suggestions. This work was supported by the National Natural Science Foundation of China (Grant No. 11971225, 12001508).

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