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Research Paper

Extension dimensions of derived and stable equivalent algebras

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ABSTRACT

The extension dimensions of an Artin algebra give a reasonable way of measuring how far an algebra is from being representation-finite. In this paper we mainly study the behavior of the extensions dimensions of algebras under different equivalences. We show that the difference of the extension dimensions of two derived equivalent algebras is bounded above by the length of the tilting complex associated with the derived equivalence, and that the extension dimension is an invariant under the stable equivalence. In addition, we provide two sufficient conditions such that the extension dimension is an invariant under particular derived equivalences.

1. Introduction

Rouquier introduced the dimension of a triangulated category in [36,37]. It is an invariant that measures how quickly the triangulated category can be built from one object. This dimension also plays an important role in the representation theory of

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Artin algebras (see [13,28,31,36]). Similar to the dimension of triangulated categories, the extension dimension of abelian categories was introduced by Beligiannis in [4]. Let A be an Artin algebra. We denote by $\text{ext.dim}(A)$ the extension dimension of the category of all finitely generated left A-modules. Beligiannis ([4]) proved that $\text{ext.dim}(A) = 0$ if and only if A is representation-finite. It means that the extension dimension of an Artin algebra gives a reasonable way of measuring how far an algebra is from being representation-finite. Though many upper bounds have been found for the extension dimension of a given Artin algebra (see [4,43,44]), the precise value of its extension dimension is very hard to directly compute. One possible strategy is to study the relationship between the extension dimensions of "nicely" related algebras. Specifically, we study in this paper the following question:

Suppose two Artin algebras A and B are derived equivalent or stably equivalent, how are the relationships of their extension dimensions?

As we known, derived equivalences play an important role in the representation theory of Artin algebras and finite groups (see [14,40]), while the Morita theory of derived categories of rings by Rickard ([33]) and the Morita theory of derived categories of differential graded algebras by Keller ([25]) provide a useful tool to understand homological properties of these equivalent algebras. Many homological invariants of derived equivalences have been discovered, for example Hochschild homology ([35]), cyclic homology ([26]), algebraic K-groups ([10]) and the number of non-isomorphic simple modules ([33]). Though derived equivalences do not always preserve homological dimensions of algebras and modules, they still can provide a useful tool to understand some homological properties of algebras. For example, the differences of global and finitistic dimensions of two derived equivalent algebras are bounded above the length of a tilting complex inducing a derived equivalence (see [11, Section 12.5(b)], [15], [32]). We hope to bound the difference of extension dimensions of derived equivalent algebras in terms of lengths of tilting complexes. Recall that the *length* of a radical complex $X^\bullet$ in $\mathscr{K}^b(A)$ is defined to be

$$\ell(X^\bullet) = \sup\{s \mid X^s \neq 0\} - \inf\{t \mid X^t \neq 0\} + 1.$$

Define the *length* of an arbitrary complex $Y^\bullet$ in $\mathscr{K}^b(A)$ to be the length of the unique radical complex that is isomorphic to $Y^\bullet$ in $\mathscr{K}^b(A)$ (Lemma 2.1). One of main results reads as the following theorem.

Theorem 1.1. (Theorem 3.4) *Let $F : \mathscr{D}^b(A) \xrightarrow{\sim} \mathscr{D}^b(B)$ be a derived equivalence between Artin algebras. Then $|\text{ext.dim}(A) - \text{ext.dim}(B)| \leq \ell(F(A)) - 1$.*

In representation theory, another important equivalence is the stable equivalence of Artin algebras. In [30], Martínez-Villa proved that stable equivalences preserve the global and dominant dimensions of algebras without nodes. Recently, Xi-Zhang ([41]) showed that the delooping levels, ϕ-dimensions and ψ-dimensions of Artin algebras are invariants of stable equivalences of algebras without nodes. Guo ([12]) showed that stable equivalences preserve the representation dimensions of Artin algebras (this was already proved by Xi in [39] for stable equivalence of Morita type). We will prove an analogous result for extension dimension.

Theorem 1.2. (Theorem 4.5) *Let A and B be stably equivalent Artin algebras. Then* $\text{ext.dim}(A) = \text{ext.dim}(B)$.

In general, one could not hope that the extension dimension is an invariant under derived equivalences. For example, a representation-finite algebra can derived equivalent to a representation-infinite algebra (see Example 3.11). In this article, we provide two sufficient conditions such that derived equivalences preserve extension dimensions.

One well-developed approach is based on the special derived equivalences. For finite dimensional selfinjective algebras, Rickard ([34]) showed that each derived equivalence induces a stable equivalence. Hu-Xi ([20]) generalized the result of Rickard by introducing a new class of derived equivalences, called almost ν-stable derived equivalences. They proved that every almost ν-stable derived equivalence always induces a stable equivalence. An application of Theorem 1.2 is the following result.

Corollary 1.3. (Corollary 4.6) *Let A and B be almost ν-stable derived equivalent finite dimensional algebras. Then* $\text{ext.dim}(A) = \text{ext.dim}(B)$.

The second approach is based on the theory of 2-term silting complexes, which generalizes classical tilting theory. Hoshino-Kato-Miyachi ([18]) proved that each 2-term silting complex $P^\bullet$ over an algebra A can induce a torsion pair $(\mathcal{T}(P^\bullet), \mathcal{F}(P^\bullet))$ in A-mod. Recently, Buan-Zhou ([7]) gave a generalization of the Brenner-Butler tilting theorem (see [5,16]), called the silting theorem. This theorem described the relations of torsion pairs between A-mod and B-mod, where $B = \text{End}_{\mathscr{D}^b(A)}(P^\bullet)$. This provides a basic framework for comparing the extension dimensions of derived equivalent algebras. Recall that a 2-term silting complex $P^\bullet$ over A is called *separating* if the induced torsion pair $(\mathcal{T}(P^\bullet), \mathcal{F}(P^\bullet))$ is split. Then one of main results can be presented as follows.

Theorem 1.4. (Theorem 3.9) *Suppose A is an Artin algebra, $P^\bullet$ a 2-term silting complex, and $B := \text{End}_{\mathscr{D}^b(A)}(P^\bullet)$. If $P^\bullet$ is separating and* $\text{id}({}_AX) \leq 1$ *for each $X \in \mathcal{F}(P^\bullet)$, then* $\text{ext.dim}(A) = \text{ext.dim}(B)$.

The paper is outlined as follows: In Section 2, we recall some basic notations, definitions and facts required in proofs. In Section 3, we compare the extension dimensions of derived equivalent algebras. We first in Section 3.1 gives a proof of Theorem 1.1. Section 3.2 then provides a sufficient condition for the extension dimension to be a derived invariant and gives several examples to illustrate the necessity of some assumptions in Theorem 1.4. In Section 4, we prove Theorem 1.2 and present an example to illustrate this main result. Corollary 1.3 is the direct consequence of Theorem 1.2.

2. Preliminaries

In this section, we shall fix some notations, and recall some definitions.

2.1. Stable equivalences and derived equivalences

Throughout this paper, k is an arbitrary but fixed commutative Artin ring. Unless stated otherwise, all algebras are Artin k-algebras with unit, and all modules are finitely generated unitary left modules; all categories will be k-categories and all functors are k-functors.

Let A be an Artin algebra. We denote by A-mod the category of all finitely generated left A-modules, and by A-ind the set of isomorphism classes of indecomposable finitely generated A-modules. All subcategories of A-mod are full, additive and closed under isomorphisms. For a class of A-modules $\mathcal{X}$, we write add$(\mathcal{X})$ for the smallest full subcategory of A-mod containing $\mathcal{X}$ and closed under finite direct sums and direct summands. When $\mathcal{X}$ consists of only one object X, we write add(X) for add$(\mathcal{X})$. In particular, add$({}_AA)$ is exactly the category of projective A-modules and also denoted by A-proj. We denote by $\mathscr{P}_A$ and $\mathscr{I}_A$ the set of isomorphism classes of indecomposable projective and injective A-modules, respectively. Let X be an A-module. If $f : P \to X$ is the projective cover of X with P projective, then the kernel of f is called the *syzygy* of X, denoted by $\Omega(X)$. Dually, if $g : X \to I$ is the injective envelope of X with I injective, then the cokernel of g is called the *cosyzygy* of X, denoted by $\Omega^{-1}(X)$. Additionally, let Ω^0 be the identity functor in A-mod and $\Omega^1 := \Omega$. Inductively, for any $n \geq 2$, define $\Omega^n(X) := \Omega^1(\Omega^{n-1}(X))$ and $\Omega^{-n}(X) := \Omega^{-1}(\Omega^{-n+1}(X))$. We denoted by pd$({}_AX)$ and id$({}_AX)$ the projective and injective dimension, respectively.

Let $A^{\rm op}$ be the opposite algebra of A, and $\mathrm{D} := \mathrm{Hom}_k(-, E(k/\mathrm{rad}(k)))$ the usual duality from A-mod to $A^{\rm op}$-mod, where rad(k) denotes the radical of k and $E(k/\mathrm{rad}(k))$ denotes the injective envelope of $k/\mathrm{rad}(k)$. The duality $\mathrm{Hom}_A(-, A)$ from A-proj to $A^{\rm op}$-proj is denoted by *, namely for each projective A-module P, the projective $A^{\rm op}$-module $\mathrm{Hom}_A(P, A)$ is written as P^*. We write ν_A for the Nakayama functor $\mathrm{DHom}_A(-, A) : A\text{-proj} \to A\text{-inj}$. An A-module X is called a *generator* if $A \in \mathrm{add}(X)$, *cogenerator* if $\mathrm{D}(A_A) \in \mathrm{add}(X)$, and *generator-cogenerator* if it is both a generator and cogenerator in A-mod.

We denoted by A-$\underline{\mathrm{mod}}$ the stable module category of A modulo projective modules. The objects are the same as the objects of A-mod, and the homomorphism set $\underline{\mathrm{Hom}}_A(X, Y)$ between X and Y is given by the quotients of $\mathrm{Hom}_A(X, Y)$ modulo those homomorphisms that factorize through a projective A-module. This category is usually called the *stable module category* of A. Dually, We denoted by A-$\overline{\mathrm{mod}}$ the stable module category of A modulo injective modules. Two algebras A and B are said to be *stably equivalent* if the two stable categories A-$\underline{\mathrm{mod}}$ and B-$\underline{\mathrm{mod}}$ are equivalent as additive categories.

Let $\mathcal{C}$ be an additive category. For two morphisms $f : X \to Y$ and $g : Y \to Z$ in $\mathcal{C}$, their composition is denoted by fg, which is a morphism from X to Z. But for two functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$ of categories, their composition is written as GF.

A complex $X^\bullet = (X^i, d_X^i)$ over $\mathcal{C}$ is a sequence of objects X^i in $\mathcal{C}$ with morphisms $d_{X^\bullet}^i : X^i \to X^{i+1}$ such that $d_{X^\bullet}^i d_{X^\bullet}^{i+1} = 0$ for all $i \in \mathbb{Z}$. We denote by $\mathscr{C}(C)$ the category of complexes over $\mathcal{C}$, and by $\mathscr{K}(\mathcal{C})$ the homotopy category of complexes over $\mathcal{C}$. If $\mathcal{C}$ is an abelian category, then we denote by $\mathscr{D}(\mathcal{C})$ the derived category of complexes over $\mathcal{C}$. Let $\mathscr{K}^b(\mathcal{C})$ be the full subcategory of $\mathscr{K}(\mathcal{C})$ consisting of bounded complexes over $\mathcal{C}$. A complex $X^\bullet$ over $\mathcal{C}$ is *cohomologically bounded* if all but finitely many cohomologies of $X^\bullet$ are zero. Let $\mathscr{D}^b(\mathcal{C})$ be the full subcategory of $\mathscr{D}(\mathcal{C})$ consisting of cohomologically bounded complexes over $\mathcal{C}$. For a given algebra A, we simply write $\mathscr{C}(A)$, $\mathscr{K}(A)$ and $\mathscr{D}(A)$ for $\mathscr{C}(A\text{-mod})$, $\mathscr{K}(A\text{-mod})$ and $\mathscr{D}(A\text{-mod})$, respectively. Similarly, we write $\mathscr{K}^b(A)$ and $\mathscr{D}^b(A)$ for $\mathscr{K}^b(A\text{-mod})$ and $\mathscr{D}^b(A\text{-mod})$, respectively. It is known that $\mathscr{K}(A)$, $\mathscr{D}(A)$, $\mathscr{K}^b(A)$ and $\mathscr{D}^b(A)$ are triangulated categories. For a complex $X^\bullet$ in $\mathscr{K}(A)$ or $\mathscr{D}(A)$, the complex $X^\bullet[1]$ is obtained from $X^\bullet$ by shifting $X^\bullet$ to the left by one degree.

Let A be an Artin algebra. A homomorphism $f : X \to Y$ of A-modules is said to be a *radical homomorphism* if, for any module Z and homomorphisms $h : Z \to X$ and $g : Y \to Z$, the composition hfg is not an isomorphism. For a complex $(X^i, d_{X^\bullet}^i)$ over A-mod, if all $d_{X^\bullet}^i$ are radical homomorphisms, then it is called a *radical complex*, which has the following properties.

Lemma 2.1. ([20, pp. 112-113]) *Let A be an Artin algebra.*

(1) *Every complex over* A-mod *is isomorphic to a radical complex in $\mathscr{K}(A)$.*

(2) *Two radical complexes $X^\bullet$ and $Y^\bullet$ are isomorphic in $\mathscr{K}(A)$ if and only if they are isomorphic in $\mathscr{C}(A)$.*

Two algebras A and B are said to be *derived equivalent* if their derived categories $\mathscr{D}^b(A)$ and $\mathscr{D}^b(B)$ are equivalent as triangulated categories. In [33], Rickard proved that A and B are derived equivalent if and only if there exists a bounded complex $T^\bullet$ of finitely generated projective A-modules such that $B \simeq \mathrm{End}_{\mathscr{D}^b(A)}(T^\bullet)$ and

(1) $\mathrm{Hom}_{\mathscr{D}^b(A)}(T^\bullet, T^\bullet[i]) = 0$ for all $i \neq 0$;

(2) $\mathscr{K}^b(A\text{-proj}) = \mathrm{thick}(T^\bullet)$, where $\mathrm{thick}(T^\bullet)$ is the smallest triangulated subcategory of $\mathscr{K}^b(A\text{-proj})$ containing $T^\bullet$ and closed under finite direct sums and direct summands.

A complex in $\mathscr{K}^b(A\text{-proj})$ satisfying the above two conditions is called a *tilting complex* over A. It is known that, given a derived equivalence $F : \mathscr{D}^b(A) \to \mathscr{D}^b(B)$, there is a unique (up to isomorphism) tilting complex $T^\bullet$ over A such that $F(T^\bullet) \simeq B$ and $F(A)$ is isomorphic in $\mathscr{D}^b(B)$ to a tilting complex over B.

Lemma 2.2. ([20, Lemma 2.1]) *Let A and B be two algebras, and let $F : \mathscr{D}^b(A) \to \mathscr{D}^b(B)$ be a derived equivalence with a quasi-inverse F^{-1}. Then $F(A)$ is isomorphic in $\mathscr{D}^b(B)$ to a complex $\bar{T}^\bullet \in \mathscr{K}^b(B\text{-proj})$ of the form*

$$0 \longrightarrow \bar{T}^0 \longrightarrow \bar{T}^1 \longrightarrow \cdots \longrightarrow \bar{T}^n \longrightarrow 0$$

for some $n \geq 0$ *if and only if* $F^{-1}(B)$ *is isomorphic in* $\mathscr{D}^b(A)$ *to a complex* $T^\bullet \in \mathscr{K}^b(A\text{-proj})$ *of the form*

$$0 \longrightarrow T^{-n} \longrightarrow \cdots \longrightarrow T^{-1} \longrightarrow T^0 \longrightarrow 0.$$

A special class of derived equivalences can be constructed by tilting modules. Recall that an A-module T is said to be a *tilting module* if T satisfied the following three conditions: (1) $\mathrm{pd}({}_AT) \leq n$, (2) $\mathrm{Ext}^i_A(T,T) = 0$ for all $i > 0$, and (3) there exists an exact sequence $0 \to A \to T_0 \to \cdots \to T_n \to 0$ in A-mod with each T_i in $\mathrm{add}({}_AT)$. Let $P^\bullet(T) : 0 \to P_n \to P_{n-1} \to \cdots \to P_0 \to 0$ be a deleted projective resolution of T. Clearly, $P^\bullet(T)$ is a tilting complex over A and $\mathrm{End}_A(T) \simeq \mathrm{End}_{\mathscr{D}^b(A)}(P^\bullet(T))$ as algebras.

2.2. Extension dimensions

In this subsection, we shall recall the definition and some results of the extension dimensions (see [4,44]).

Let A be an Artin algebra. We denote by A-mod the category of all finitely generated left A-modules. For a class $\mathcal{T}$ of A-modules, we denote by $\mathrm{add}(\mathcal{T})$ the smallest full subcategory of A-mod containing $\mathcal{T}$ and closed under finite direct sums and direct summands. When $\mathcal{T}$ consists of only one object T, we write $\mathrm{add}(T)$ for $\mathrm{add}(\mathcal{T})$. Let $\mathcal{T}_1, \mathcal{T}_2, \cdots, \mathcal{T}_n$ be subcategories of A-mod. Define

$$\begin{aligned}
&\mathcal{T}_1 \bullet \mathcal{T}_2 \\
&\quad := \mathrm{add}(\{X \in A\text{-mod} \mid \text{there exists an exact sequence } 0 \longrightarrow T_1 \longrightarrow X \longrightarrow T_2 \longrightarrow 0 \\
&\qquad\qquad \text{in } A\text{-mod with } T_1 \in \mathcal{T}_1 \text{ and } T_2 \in \mathcal{T}_2\}) \\
&\quad = \{X \in A\text{-mod} \mid \text{there exists an exact sequence } 0 \longrightarrow T_1 \longrightarrow X \oplus X' \longrightarrow T_2 \longrightarrow 0 \\
&\qquad\qquad \text{in } A\text{-mod for some } A\text{-module } X' \text{ with } T_1 \in \mathcal{T}_1 \text{ and } T_2 \in \mathcal{T}_2\}.
\end{aligned}$$

The operation $\bullet$ is associative. Inductively, define

$$\begin{aligned}
&\mathcal{T}_1 \bullet \mathcal{T}_2 \bullet \cdots \bullet \mathcal{T}_n \\
&\quad := \mathrm{add}(\{X \in A\text{-mod} \mid \text{there exists an exact sequence } 0 \longrightarrow T_1 \longrightarrow X \longrightarrow T_2 \longrightarrow 0 \\
&\qquad \text{in } A\text{-mod with } T_1 \in \mathcal{T}_1 \text{ and } T_2 \in \mathcal{T}_2 \bullet \cdots \bullet \mathcal{T}_n\}).
\end{aligned}$$

Thus $X \in \mathcal{T}_1 \bullet \mathcal{T}_2 \bullet \cdots \bullet \mathcal{T}_n$ if and only if there exist the following exact sequences

$$\begin{cases}
0 \longrightarrow T_1 \longrightarrow X \oplus X_1' \longrightarrow X_2 \longrightarrow 0, \\
0 \longrightarrow T_2 \longrightarrow X_2 \oplus X_2' \longrightarrow X_3 \longrightarrow 0, \\
\qquad\qquad \vdots \\
0 \longrightarrow T_{n-1} \longrightarrow X_{n-1} \oplus X_{n-1}' \longrightarrow X_n \longrightarrow 0,
\end{cases}$$

for some A-modules X_i' such that $T_i \in \mathcal{T}_i$ and $X_{i+1} \in \mathcal{T}_{i+1} \bullet \mathcal{T}_{i+2} \bullet \cdots \bullet \mathcal{T}_n$ for $1 \leq i \leq n-1$.

For a subcategory $\mathcal{T}$ of A-mod, set $[\mathcal{T}]_0 := \{0\}$, $[\mathcal{T}]_1 := \mathrm{add}(\mathcal{T})$, $[\mathcal{T}]_n = [\mathcal{T}]_1 \bullet [\mathcal{T}]_{n-1}$ for any $n \geq 2$. If $T \in A$-mod, we write $[T]_n$ instead of $[\{T\}]_n$.

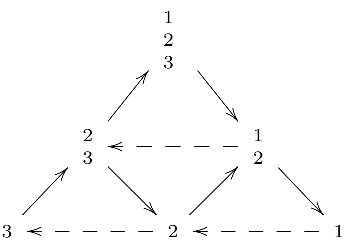
It is worth noting that, in general, we have $\mathcal{T}_1 \bullet \mathcal{T}_2 \neq \mathcal{T}_1 \diamond \mathcal{T}_2$ where

$$\mathcal{T}_1 \diamond \mathcal{T}_2 := \{X \in A\text{-mod} \mid \text{there exists an exact sequence } 0 \longrightarrow T_1 \longrightarrow X \longrightarrow T_2 \longrightarrow 0 \text{ in } A\text{-mod with } T_1 \in \mathcal{T}_1 \text{ and } T_2 \in \mathcal{T}_2\}.$$

For example, let A be a finite dimensional algebra over a field k given by the following quiver Q:

$$1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3.$$

The Aulander-Reiten quiver of A-mod is as follows:



Let $\mathcal{T}_1 := \mathrm{add}(\begin{smallmatrix}2\\3\end{smallmatrix})$ and $\mathcal{T}_2 := \mathrm{add}(\begin{smallmatrix}1\\2\end{smallmatrix})$. It follows from the following exact sequence

$$0 \longrightarrow \begin{smallmatrix}2\\3\end{smallmatrix} \longrightarrow \begin{smallmatrix}1\\2\\3\end{smallmatrix} \oplus 2 \longrightarrow \begin{smallmatrix}1\\2\end{smallmatrix} \longrightarrow 0$$

that the simple module $S(2) := 2 \in \mathcal{T}_1 \bullet \mathcal{T}_2$. Suppose $S(2) \in \mathcal{T}_1 \diamond \mathcal{T}_2$. Then we have the following short exact sequence

$$0 \longrightarrow T_1 \longrightarrow S(2) \longrightarrow T_2 \longrightarrow 0 \tag{2.1}$$

such that $T_1 \in \mathcal{T}_1$ and $T_2 \in \mathcal{T}_2$. The sequence (2.1) is split since $S(2)$ is simple, and we get that $S(2) \cong T_1 \oplus T_2$. Moreover, we have $S(2) \cong T_1 \in \mathcal{T}_1$ or $S(2) \cong T_2 \in \mathcal{T}_2$. This is a contradiction. That is, we found a module in $\mathcal{T}_1 \bullet \mathcal{T}_2$, but it's not in $\mathcal{T}_1 \diamond \mathcal{T}_2$.

Definition 2.3. ([4]) The extension dimension of A-mod is defined to be

$$\mathrm{ext.dim}(A) := \inf\{n \geq 0 \mid A\text{-mod} = [T]_{n+1} \text{ with } T \in A\text{-mod}\}.$$

Lemma 2.4. *Let A be an Artin algebra.*

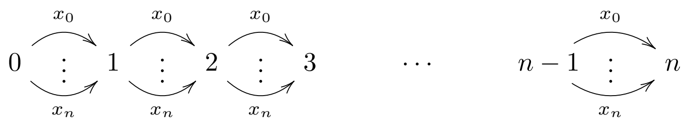
(1) ([4, Example 1.6]) *A is representation finite if and only if* $\operatorname{ext.dim}(A) = 0$.

(2) ([4, Example 1.6]) $\operatorname{ext.dim}(A) \leq \ell\ell(A) - 1$, *where $\ell\ell(A)$ stands for the Loewy length of A.*

(3) ([44, Corollary 3.6]) $\operatorname{ext.dim}(A) \leq \operatorname{gl.dim}(A)$, *where* $\operatorname{gl.dim}(A)$ *stands for the global dimension of A.*

(4) ([43, Corollary 3.15]) $\operatorname{ext.dim}(A) \leq \ell\ell^{\infty}(A) + \max\{\operatorname{pd}({}_AS) \mid {}_AS$ *is simple with* $\operatorname{pd}({}_AS) < \infty\}$, *where $\ell\ell^{\infty}(A)$ stands for the infinite-layer length of A* ([22,23]).

Example 2.5. Let A be the Beilinson algebra kQ/I with quiver Q



and relations $I = (x_i x_j - x_j x_i)$ for $0 \leq i, j \leq n$. By [43, Example 3.4], we know that $\operatorname{ext.dim}(A) = n$. We see that the extension dimension may be very large.

Definition 2.6. ([24, Defition 4.5(2)]) Let M be an A-module. Then the *weak M-resolution dimension* of an A-module X is defined to be

$$
\begin{aligned}
&M\text{- w.resol.dim}(X)\\
&:= \inf\{n \in \mathbb{N} \mid \text{ there is an exact sequence } 0 \to M_n \to M_{n-1} \to \cdots \to M_0 \to X' \to 0\\
&\qquad\qquad \text{with all } M_i \in \operatorname{add}(M) \text{ and } X \in \operatorname{add}(X')\}\\
&= \inf\{n \in \mathbb{N} \mid \text{ there is an exact sequence } 0 \to M_n \to M_{n-1} \to \cdots \to M_0 \to X \oplus Y \to 0\\
&\qquad\qquad \text{for some } A\text{-module } Y \text{ with all } M_i \in \operatorname{add}(M)\}.
\end{aligned}
$$

Here we set $\inf \varnothing = \infty$; and the *weak M-resolution dimension* of algebra A is defined to be

$$M\text{-w.resol.dim}(A) := \sup\{M\text{-w.resol.dim}(X) \mid X \in A\text{-mod}\};$$

the *weak resolution dimension* of algebra A is defined to be

$$\text{w.resol.dim}(A) := \inf\{M\text{-w.resol.dim}(A) \mid M \in A\text{-mod}\}.$$

Dually, we can define the *weak M-coresolution dimension*, the *weak M-coresolution dimension* of A and the *weak coresolution dimension* of A. By [24, Definition 4.5], we know that the weak resolution dimension of A and the weak coresolution dimension of A are the same.

Recall that a module is called *basic* if it is a direct sum of non-isomorphic indecomposable modules. Then each module M admits a unique decomposition (up to isomorphism) $M \simeq M_b \oplus M_0$ such that M_b is basic and $M_0 \in \mathrm{add}(M_b)$. Then M-w.resol.dim$(A) = M_b$-w.resol.dim(A) and

$$\text{w.resol.dim}(A) = \min\{M\text{-w.resol.dim}(A) \mid M \in A\text{-mod and } M \text{ is basic}\}.$$

In the following lemma, we mention a few basic properties of weak resolution dimensions.

Lemma 2.7. *Let M be an A-module. Then*

(1) *For A-modules $X_i, 1 \leq i \leq n$, we have*

$$M\text{-w.resol.dim}(\bigoplus_{i=1}^{n} X_i) = \sup\{M\text{-w.resol.dim}(X_i) \mid 1 \leq i \leq n\}.$$

(2) M-w.resol.dim$(A) = \sup\{M\text{-w.resol.dim}(X) \mid X \in A\text{-ind}\}$, *where* A-ind *stands for the set of isomorphism classes of indecomposable finitely generated A-modules.*

(3) *Fix an A-module M_0, we have* $(M \oplus M_0)$-w.resol.dim$(A) \leq M$-w.resol.dim(A). *In particular,*

$$\text{w.resol.dim}(A) = \inf\{(M' \oplus M_0)\text{-w.resol.dim}(A) \mid M' \in A\text{-mod } \textit{and } M' \textit{ is basic}\}.$$

In [44, Theorem 3.5], Zheng-Ma-Huang argued that the weak resolution dimension and the extension dimension of an Artin algebra coincide. We think that the proof of [44, Theorem 3.5] is incomplete since the middle term of the short exact sequence has missing direct summand (for more details, see the proof of [44, Theorem 3.5]). For convenience, the full proof is given here. To prove the result, we need the following lemma.

Lemma 2.8. *Let A be an Artin algebra.*

(1) (see [42, Lemma 4.6]) *Given the exact sequence* $0 \to X \to Y \to Z \to 0$ *in* A-mod, *we can get the following exact sequence*

$$0 \longrightarrow \Omega^{i+1}(Z) \longrightarrow \Omega^{i}(X) \oplus P_i \longrightarrow \Omega^{i}(Y) \longrightarrow 0$$

for some projective module P_i in A-mod, *where $i \geq 0$.*

(2) ([43, Lemma 3.5]) *Let* $0 \to M_n \to M_{n-1} \to \cdots \to M_0 \to X \to 0$ *be an exact sequence in* A-mod. *Then*

$$X \in [M_0]_1 \bullet [\Omega^{-1}(M_1)]_1 \bullet \cdots \bullet [\Omega^{-n+1}(M_{n-1})]_1 \bullet [\Omega^{-n}(M_n)]_1 \subseteq [\bigoplus_{i=0}^{n} \Omega^{-i}(M_i)]_{n+1}.$$

Lemma 2.9. *For an Artin algebra A, we have* w.resol.dim$(A) = $ ext.dim(A).

Proof. Suppose ext.dim$(A) = n$. By Definition 2.3, there exists an A-module T such that

$$A\text{-mod} = [T]_{n+1}.$$

Then, for an A-module X_0, there are short exact sequences

$$\begin{cases} 0 \longrightarrow T_0 \longrightarrow X_0 \oplus X_0' \longrightarrow X_1 \longrightarrow 0, \\ 0 \longrightarrow T_1 \longrightarrow X_1 \oplus X_1' \longrightarrow X_2 \longrightarrow 0, \\ \qquad \vdots \\ 0 \longrightarrow T_{n-1} \longrightarrow X_{n-1} \oplus X_{n-1}' \longrightarrow X_n \longrightarrow 0. \end{cases}$$

in A-mod for some A-modules X_i' such that $T_i \in [T]_1$ and $X_{i+1} \in [T]_{n-i}$ for each $0 \le i \le n-1$. Furthermore, we have short exact sequences

$$\begin{cases} 0 \longrightarrow T_0 \longrightarrow X_0 \oplus (\bigoplus_{j=0}^{n-1} X_j') \longrightarrow X_1 \oplus (\bigoplus_{j=1}^{n-1} X_j') \to 0, \\ 0 \longrightarrow T_1 \longrightarrow X_1 \oplus (\bigoplus_{j=1}^{n-1} X_j') \longrightarrow X_2 \oplus (\bigoplus_{j=2}^{n-1} X_j') \to 0, \\ \qquad \vdots \\ 0 \to T_{n-2} \to X_{n-2} \oplus (\bigoplus_{j=n-2}^{n-1} X_j') \to X_{n-1} \oplus X_{n-1}' \to 0, \\ 0 \to T_{n-1} \longrightarrow X_{n-1} \oplus X_{n-1}' \longrightarrow X_n \longrightarrow 0. \end{cases} \tag{2.2}$$

By Lemma 2.8(1) and the short exact sequences (2.2), we have the following short exact sequences

$$\begin{cases} 0 \to \Omega^1(X_1 \oplus (\bigoplus_{j=1}^{n-1} X_j')) \longrightarrow T_0 \oplus P_0 \longrightarrow X_0 \oplus (\bigoplus_{j=0}^{n-1} X_j') \longrightarrow 0, \\ 0 \to \Omega^2(X_2 \oplus (\bigoplus_{j=2}^{n-1} X_j')) \longrightarrow \Omega^1(T_1) \oplus P_1 \longrightarrow \Omega^1(X_1 \oplus (\bigoplus_{j=1}^{n-1} X_j')) \longrightarrow 0, \\ \qquad \vdots \\ 0 \to \Omega^{n-1}(X_{n-1} \oplus X_{n-1}') \to \Omega^{n-2}(T_{n-2}) \oplus P_{n-2} \to \Omega^{n-2}(X_{n-2} \oplus (\bigoplus_{j=n-2}^{n-1} X_j')) \to 0, \\ 0 \longrightarrow \Omega^n(X_n) \longrightarrow \Omega^{n-1}(T_{n-1}) \oplus P_{n-1} \longrightarrow \Omega^{n-1}(X_{n-1} \oplus X_{n-1}') \longrightarrow 0 \end{cases} \tag{2.3}$$

where all ${}_AP_i$ are projective. By the short exact sequences (2.3), we get the following long short sequence

$$0 \longrightarrow \Omega^n(X_n) \longrightarrow \Omega^{n-1}(T_{n-1}) \oplus P_{n-1} \longrightarrow \cdots \longrightarrow \Omega^1(T_1) \oplus P_1 \longrightarrow T_0 \oplus P_0$$
$$\longrightarrow X_0 \oplus (\bigoplus_{j=i}^{n-1} X_0') \longrightarrow 0. \tag{2.4}$$

Set $N := (\bigoplus_{0\leq i\leq n} \Omega^i(T)) \oplus A$. It follows from $X_n \in [T]_1$, $P_i \in \mathrm{add}({}_AA)$ and $T_i \in [T]_1$ that $\Omega^n(X_n), P_i, \Omega^i(T_i) \in \mathrm{add}({}_AN)$ for $0 \leq i \leq n-1$. By Definition 2.6 and the long exact sequence (2.4), we get

$$N\text{-}\,\mathrm{w.resol.dim}(X_0) \leq n \quad \text{and} \quad N\text{-}\,\mathrm{w.resol.dim}(A) \leq n.$$

By Definition 2.6, we get

$$\mathrm{w.resol.dim}(A) \leq N\text{-}\,\mathrm{w.resol.dim}(A) \leq n = \mathrm{ext.dim}(A).$$

Conversely, suppose $\mathrm{w.resol.dim}(A) = m$. By Definition 2.6, there exists an A-module M such that, for any A-module X, there is an exact sequence

$$0 \longrightarrow M_m \longrightarrow M_{m-1} \longrightarrow \cdots \longrightarrow M_0 \longrightarrow X \oplus Y \longrightarrow 0$$

in A-mod for some A-module Y such that $M_i \in \mathrm{add}(M)$ for $0 \leq i \leq m$. By Lemma 2.8(2), we have $X \oplus Y \in [\bigoplus_{i=0}^m \Omega^{-i}(M_i)]_{m+1}$ and $X \in [\bigoplus_{i=0}^m \Omega^{-i}(M_i)]_{m+1}$. Then A-mod $= [\bigoplus_{i=0}^m \Omega^{-i}(M_i)]_{m+1}$. By Definition 2.3,

$$\mathrm{ext.dim}(A) \leq m = \mathrm{w.resol.dim}(A).$$

Thus we have $\mathrm{w.resol.dim}(A) = \mathrm{ext.dim}(A)$. □

3. Derived equivalences

In this section, we discuss the relationships of the extension dimensions of two derived equivalent algebras. In the first subsection, we get how much extension dimensions can vary under derived equivalences. The second subsection provides a sufficient condition such that two derived equivalent algebras have the same extension dimensions.

3.1. Variance of extension dimensions under derived equivalences

In this subsection, we first review some of the basic facts and conclusions, as detailed in reference [19].

Definition 3.1. A derived equivalence $F : \mathscr{D}^b(A) \to \mathscr{D}^b(B)$ is called *nonnegative* if

(1) $F(X)$ is isomorphic to a complex with zero cohomology in all negative degrees for all $X \in A$-mod; and (2) $F(P)$ is isomorphic to a complex in $\mathscr{K}^b(B\text{-proj})$ with zero terms in all negative degrees for all $P \in A$-proj.

Lemma 3.2. ([19, Lemma 4.2]) *Let $F : \mathscr{D}^b(A) \to \mathscr{D}^b(B)$ be a derived equivalence and $T^\bullet \in \mathscr{K}^b(A\text{-proj})$ be the radical tilting complex over A such that $F(T^\bullet) \simeq B$ in $\mathscr{D}^b(B)$. Then F is nonnegative if and only if the tilting complex $T^\bullet$ is isomorphic to a complex with zero terms in all positive degrees. In particular, $F[i]$ is nonnegative for sufficiently small i.*

For every nonnegative derived equivalence F, Hu-Xi (see [20, Section 3]) construct a functor $\overline{F} : A\text{-}\underline{\text{mod}} \to B\text{-}\underline{\text{mod}}$, which is called the *stable functor* of F. This stable functor has the following properties.

Lemma 3.3. (see [20] or [19, Section 4]) (1) *Let i be a nonnegative integer. Then i-th syzygy functor $\Omega^i_A : A\text{-}\underline{\text{mod}} \to A\text{-}\underline{\text{mod}}$ is a stable functor of the derived equivalence $[-i] : \mathscr{D}^b(A) \to \mathscr{D}^b(A)$, that is, $\overline{[-i]} \simeq \Omega^i_A$ as additive functors. In particular, the stable functor of identity functor on $\mathscr{D}^b(A)$ is isomorphic to the identity functor on $A\text{-}\underline{\text{mod}}$.*

(2) *Let $F : \mathscr{D}^b(A) \to \mathscr{D}^b(B)$ and $G : \mathscr{D}^b(B) \to \mathscr{D}^b(C)$ be two nonnegative derived equivalences. Then the functors $\overline{G} \circ \overline{F}$ and $\overline{GF}$ are isomorphic.*

(3) *Let $F : \mathscr{D}^b(A) \to \mathscr{D}^b(B)$ be a nonnegative derived equivalence. Suppose that*

$$0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$$

is an exact sequence in A-mod. Then there is an exact sequence

$$0 \longrightarrow \overline{F}(X) \longrightarrow \overline{F}(Y) \oplus Q \longrightarrow \overline{F}(Z) \longrightarrow 0$$

in B-mod for some projective B-module Q.

Theorem 3.4. *Let $F : \mathscr{D}^b(A) \xrightarrow{\sim} \mathscr{D}^b(B)$ be a derived equivalence between Artin algebras. Then $|\operatorname{ext.dim}(A) - \operatorname{ext.dim}(B)| \le \ell(F(A)) - 1$.*

Proof. Let G be the quasi-inverse F, and let $T^\bullet \in \mathscr{K}^b(A\text{-proj})$ and $\bar{T}^\bullet \in \mathscr{K}^b(B\text{-proj})$ be the radical tilting complexes such that $F(T^\bullet) \simeq B$ in $\mathscr{D}^b(B)$ and $G(\bar{T}^\bullet) \simeq A$ in $\mathscr{D}^b(A)$, respectively. Then $T^\bullet \simeq G(B)$ in $\mathscr{D}^b(A)$ and $\bar{T}^\bullet \simeq F(A)$ in $\mathscr{D}^b(B)$.

Set $n := \ell(F(A)) - 1$. Then $\ell(\bar{T}) = \ell(F(A)) = n + 1$. By applying the shift functor, we can assume that $\bar{T}^\bullet \in \mathscr{K}^b(B\text{-proj})$ is of the form

$$0 \longrightarrow \bar{T}^0 \longrightarrow \bar{T}^1 \longrightarrow \cdots \longrightarrow \bar{T}^n \longrightarrow 0.$$

By Lemma 2.2, $T^\bullet \in \mathscr{K}^b(A\text{-proj})$ is of the form

$$0 \longrightarrow T^{-n} \longrightarrow \cdots \longrightarrow T^{-1} \longrightarrow T^0 \longrightarrow 0.$$

By Lemma 3.2, we know that F and $G[-n]$ are nonnegative.

By Lemma 2.9, we obtain that the extension dimension and the weak resolution dimension of an algebra are the same. We have to show $|\,\mathrm{w.resol.dim}(A) - \mathrm{w.resol.dim}(B)| \leq n$. We first show $\mathrm{w.resol.dim}(B) \leq \mathrm{w.resol.dim}(A) + n$.

Set $t := \mathrm{w.resol.dim}(A)$. By Definition 2.6, there exists an A-module M such that $\mathrm{w.resol.dim}(A) = M\text{-}\,\mathrm{w.resol.dim}(A) = t$. We claim

$$(\overline{F}(M) \oplus B \oplus D(B_B))\text{-}\,\mathrm{w.resol.dim}(B) \leq t + n.$$

Indeed, for $Y \in B$-mod, $\overline{G[-n]}(Y) \in A$-mod. By Definition 2.6, we have the following exact sequence

$$0 \longrightarrow M_t \xrightarrow{f_t} M_{t-1} \xrightarrow{f_{t-1}} \cdots \xrightarrow{f_2} M_1 \xrightarrow{f_1} M_0 \xrightarrow{f_0} \overline{G[-n]}(Y) \oplus Z \longrightarrow 0$$

in A-mod for some A-module Z with $M_i \in \mathrm{add}(_AM)$ for each i. Then there are short exact sequences

$$\begin{cases} 0 \to K_1 \to M_0 \to \overline{G[-n]}(Y) \oplus Z \to 0, \\ 0 \to K_2 \to M_1 \to K_1 \to 0, \\ \quad\vdots \\ 0 \to K_{t-1} \to M_{t-2} \to K_{t-2} \to 0, \\ 0 \to K_t \to M_{t-1} \to K_{t-1} \to 0, \end{cases} \tag{3.1}$$

where K_{j+1} is the kernel of f_j for $0 \leq j \leq t-1$. By Lemma 3.3(3), applying the stable functor $\overline{F}$ of F to the short exact sequences (3.1) in A-mod, we can get the following short exact sequences

$$\begin{cases} 0 \to \overline{F}(K_1) \to \overline{F}(M_0) \oplus Q_0 \to \overline{F}(\overline{G[-n]}(Y) \oplus Z) \to 0, \\ 0 \to \overline{F}(K_2) \to \overline{F}(M_1) \oplus Q_1 \to \overline{F}(K_1) \to 0, \\ \quad\vdots \\ 0 \to \overline{F}(K_{t-1}) \to \overline{F}(M_{t-2}) \oplus Q_{t-2} \to \overline{F}(K_{t-2}) \to 0, \\ 0 \to \overline{F}(K_t) \to \overline{F}(M_{t-1}) \oplus Q_{t-1} \to \overline{F}(K_{t-1}) \to 0, \end{cases} \tag{3.2}$$

in B-mod for some projective B-modules Q_i for $0 \leq i \leq t-1$. Combine the short exact sequences (3.2), we get the following long exact sequence

$$0 \longrightarrow \overline{F}(M_t) \longrightarrow \overline{F}(M_{t-1}) \oplus Q_{t-1} \longrightarrow \cdots \longrightarrow \overline{F}(M_0) \oplus Q_0 \longrightarrow \overline{F}(\overline{G[-n]}(Y) \oplus Z) \longrightarrow 0 \tag{3.3}$$

in B-mod. On the other hand, we have the following isomorphisms in B-$\underline{\text{mod}}$

$$\begin{aligned}\overline{F}(\overline{G[-n]}(Y) \oplus Z) &\simeq \overline{F}(\overline{G[-n]}(Y)) \oplus \overline{F}(Z)\\ &\simeq (\overline{F \circ G} \circ \overline{[-n]}(Y)) \oplus \overline{F}(Z) \quad \text{(by Lemma 3.3(2))}\\ &\simeq \overline{[-n]}(Y) \oplus \overline{F}(Z) \quad \text{(by Lemma 3.3(1))}\\ &\simeq \Omega_B^n(Y) \oplus \overline{F}(Z) \quad \text{(by Lemma 3.3(1)).}\end{aligned}$$

By [17, Theorem 2.2], there are projective B-modules Q and Q' such that

$$\overline{F}(\overline{G[-n]}(Y) \oplus Z) \oplus Q' \simeq \Omega_B^n(Y) \oplus \overline{F}(Z) \oplus Q$$

as B-modules. Note that there exist the following two exact sequences

$$0 \longrightarrow \Omega_B^n(Y) \longrightarrow P_{n-1} \longrightarrow P_{n-2} \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow Y \longrightarrow 0, \text{ and} \tag{3.4}$$

$$0 \longrightarrow \overline{F}(Z) \longrightarrow J^0 \longrightarrow J^1 \longrightarrow \cdots \longrightarrow J^{n-2} \longrightarrow J^{n-1} \longrightarrow \Omega^{-n}(\overline{F}(Z)) \longrightarrow 0 \tag{3.5}$$

in B-mod, where $P_j \in \text{add}({}_BB)$ and all $J^j \in \text{add}({}_BD(B_B))$ for $0 \leq j \leq n-1$. By the long exact sequences (3.3), (3.4), and (3.5), we get the following long exact sequence

$$0 \longrightarrow \overline{F}(M_t) \longrightarrow \overline{F}(M_{t-1}) \oplus Q_{t-1} \to \cdots \to \overline{F}(M_1) \oplus Q_1 \longrightarrow \overline{F}(M_0) \oplus Q_0 \oplus Q' \longrightarrow P_{n-1} \oplus J^0 \oplus Q \longrightarrow P_{n-2} \oplus J^1 \to \cdots \to P_0 \oplus J^{n-1} \longrightarrow Y \oplus \Omega^{-n}(\overline{F}(Z)) \longrightarrow 0. \tag{3.6}$$

It follows from $M_i \in \text{add}({}_AM)$ that $\overline{F}(M_i) \in \text{add}({}_B\overline{F}(M))$ for $0 \leq i \leq t$. Also $Q_i, Q, Q' \in \text{add}({}_BB)$ for $0 \leq i \leq t-1$, $P_j \in \text{add}({}_BB)$ and $J^j \in \text{add}(\mathrm{D}(B_B))$ for $0 \leq j \leq n-1$. By Definition 2.6 and the long exact sequence (3.6), we get

$$(\overline{F}(M) \oplus B \oplus \mathrm{D}(B_B))\text{- w.resol.dim}(Y) \leq t+n \quad \text{and}$$

$$\text{w.resol.dim}(B) \leq t+n = \text{w.resol.dim}(A) + n.$$

Similarly, we have $\text{w.resol.dim}(A) \leq \text{w.resol.dim}(B) + n$. Thus $|\text{w.resol.dim}(A) - \text{w.resol.dim}(B)| \leq n$. By Lemma 2.9, we get $|\text{ext.dim}(A) - \text{ext.dim}(B)| \leq n$. □

As an immediate consequence of Theorem 3.4, we have

Corollary 3.5. *Let A be an Artin algebra, T be a tilting A-module and $B = \text{End}_A(T)$, Then we have* $|\text{ext.dim}(A) - \text{ext.dim}(B)| \leq \text{pd}({}_AT)$.

Proof. Let $n := \mathrm{pd}({}_AT)$. We have the following minimal projective resolution of ${}_AT$

$$0 \longrightarrow P_n \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow T \longrightarrow 0.$$

Moreover, the following complex

$$P^\bullet(T) : 0 \longrightarrow P_n \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow 0$$

is a tilting complex in $\mathscr{D}^b(A)$ and $B \simeq \mathrm{End}_{\mathscr{D}^b(A)}(P^\bullet(T))$. Thus, by Theorem 3.4, we get the result. □

3.2. The extension invariants of 2-term silting complexes

Let A be an Artin algebra. Recall that a pair $(\mathcal{U}, \mathcal{V})$ of full subcategories of A-mod is called a *torsion pair* [9] if the following conditions are satisfied:

(1) $\mathrm{Hom}_A(\mathcal{U}, V) = 0$ if and only if $V \in \mathcal{V}$;

(2) $\mathrm{Hom}_A(U, \mathcal{V}) = 0$ if and only if $U \in \mathcal{U}$.

The subcategory $\mathcal{U}$ is called the *torsion class* and the subcategory $\mathcal{V}$ is called the *torsion-free class*. It is known (see [1, Proposition 1.1]) that a subcategory $\mathcal{U}$ (respectively, $\mathcal{V}$) of A-modules is a torsion class (respectively, torsion-free class) of a torsion pair in A-mod if and only if $\mathcal{U}$ (respectively, $\mathcal{V}$) is closed under images (respectively, submodules), direct sums and extensions. A torsion pair $(\mathcal{U}, \mathcal{V})$ is called *split* (or sometimes splitting) if each indecomposable A-module lies either in $\mathcal{U}$ or in $\mathcal{V}$.

Recall that a complex $P^\bullet = (P^i)$ is 2-term if $P^i = 0$ for $i \neq -1, 0$. A 2-term complex $P^\bullet \in \mathscr{K}^b(A\text{-proj})$ is said to be *silting* [27] if $\mathrm{Hom}_{\mathscr{K}^b(A\text{-proj})}(P^\bullet, P^\bullet[1]) = 0$ and $\mathrm{thick}(P^\bullet) = \mathscr{K}^b(A\text{-proj})$, where $\mathrm{thick}(P^\bullet)$ is the smallest triangulated subcategory of $\mathscr{K}^b(A\text{-proj})$ containing $P^\bullet$ and closed under finite direct sums and direct summands. If, in addition, $\mathrm{Hom}_{\mathscr{K}^b(A\text{-proj})}(P^\bullet, P^\bullet[-1]) = 0$, then it is easy to see that $P^\bullet$ is *tilting*.

Let $P^\bullet$ be a 2-term silting complex in $\mathscr{K}^b(A\text{-proj})$, and consider the following two full subcategories of A-mod given by

$$\mathcal{T}(P^\bullet) = \{U \in A\text{-mod} \mid \mathrm{Hom}_{\mathscr{D}^b(A)}(P^\bullet, U[1]) = 0\}, \text{ and}$$
$$\mathcal{F}(P^\bullet) = \{V \in A\text{-mod} \mid \mathrm{Hom}_{\mathscr{D}^b(A)}(P^\bullet, V) = 0\}.$$

The following lemma is a generalization of the Brenner-Butler tilting theorem (see [5,16]) to 2-term silting complexes.

Lemma 3.6. ([7]) *Let $P^\bullet$ be a 2-term silting complex in $\mathscr{K}^b(A\text{-proj})$, and let $B = \mathrm{End}_{\mathscr{D}^b(A)}(P^\bullet)$.*

(1) *Let $\mathcal{C}(P^\bullet) := \{W^\bullet \in \mathscr{D}^b(A) \mid \mathrm{Hom}_{\mathscr{D}^b(A)}(P^\bullet, W^\bullet[i]) = 0, \ \forall i \neq 0\}$. Then $\mathcal{C}(P^\bullet)$ is an abelian category and the short exact sequences in $\mathcal{C}(P^\bullet)$ are precisely the triangles in $\mathscr{D}^b(A)$ all of whose terms are objects in $\mathcal{C}(P^\bullet)$.*

(2) *The pairs* $(\mathcal{T}(P^\bullet), \mathcal{F}(P^\bullet)$ *and* $(\mathcal{F}(P^\bullet)[1], \mathcal{T}(P^\bullet))$ *are torsion pairs in* A-mod *and* $\mathcal{C}(P^\bullet)$*, respectively.*

(3) $\mathrm{Hom}_{\mathscr{D}^b(A)}(P^\bullet, -) : \mathcal{C}(P^\bullet) \to B\text{-mod}$ *is an equivalence of abelian categories. In particular, if* $P^\bullet$ *is a tilting complex, then there exists an equivalence of triangulated categories* $F : \mathscr{D}^b(A) \to \mathscr{D}^b(B)$ *such that* $F^{-1}(B) \simeq P^\bullet$ *and* $F(W^\bullet) \simeq \mathrm{Hom}_{\mathscr{D}^b(A)}(P^\bullet, W^\bullet)$ *for any* $W^\bullet \in \mathcal{C}(P^\bullet)$.

(4) *There is a triangle*

$$A \longrightarrow P_1^\bullet \xrightarrow{f} P_0^\bullet \longrightarrow A[1]$$

in $\mathscr{K}^b(A\text{-proj})$ *with* $P_1^\bullet,\ P_0^\bullet \in \mathrm{add}(P^\bullet)$.

Consider the following 2*-term complex* $Q^\bullet$ *in* $\mathscr{K}^b(B\text{-proj})$.

$$0 \longrightarrow \mathrm{Hom}_{\mathscr{D}^b(A)}(P^\bullet, P_1^\bullet) \xrightarrow{\mathrm{Hom}_{\mathscr{D}^b(A)}(P^\bullet, f)} \mathrm{Hom}_{\mathscr{D}^b(A)}(P^\bullet, P_0^\bullet) \longrightarrow 0.$$

(5) $Q^\bullet$ *is a* 2*-term silting complex in* $\mathscr{K}^b(B\text{-proj})$.

(6) *There is an algebra epimorphism* $\Phi : A \to \bar{A} := \mathrm{End}_{\mathscr{D}^b(B)}(Q^\bullet)$*. Moreover,* Φ *is an isomorphism if and only if* $P^\bullet$ *is tilting.*

(7) *Let* $\Phi_* : \bar{A}\text{-mod} \to A\text{-mod}$ *be the inclusion functor induced by* Φ *in* (6)*. Then the functor* $\mathrm{Hom}_{\mathscr{D}^b(A)}(P^\bullet, -) : \mathcal{T}(P^\bullet) \to \mathcal{F}(Q^\bullet)$ *is an equivalence with quasi-inverse* $\Phi_*\mathrm{Hom}_{\mathscr{D}^b(B)}(Q^\bullet, -[1])$*; the functor* $\mathrm{Hom}_{\mathscr{D}^b(A)}(P^\bullet, -[1]) : \mathcal{F}(P^\bullet) \to \mathcal{T}(Q^\bullet)$ *is an equivalence with quasi-inverse* $\Phi_*\mathrm{Hom}_{\mathscr{D}^b(B)}(Q^\bullet, -)$.

$$\mathcal{T}(P^\bullet) \underset{\Phi_*\mathrm{Hom}_{\mathscr{D}^b(B)}(Q^\bullet,-[1])}{\overset{\mathrm{Hom}_{\mathscr{D}^b(A)}(P^\bullet,-)}{\rightleftarrows}} \mathcal{F}(Q^\bullet) \quad \textit{and} \quad \mathcal{F}(P^\bullet) \underset{\Phi_*\mathrm{Hom}_{\mathscr{D}^b(B)}(Q^\bullet,-)}{\overset{\mathrm{Hom}_{\mathscr{D}^b(A)}(P^\bullet,-[1])}{\rightleftarrows}} \mathcal{T}(Q^\bullet).$$

Proof. Note that (1)-(3) is from [18], (4) is from [38, Theorem 3.5], (5) and (6) can be from [6, Propositions A.3 and A.5], and (7) is from [7, Theorem 1.1]. □

In the following, the symbol $Q^\bullet$ always denotes the induced complex $Q^\bullet$. It is a 2-term silting complex in $\mathscr{K}^b(B\text{-proj})$.

Definition 3.7. ([7, Definition 5.4]) Let $P^\bullet$ be a 2-term silting complex in $\mathscr{K}^b(A\text{-proj})$.

(1) $P^\bullet$ is called *separating* if the induced torsion pair $(\mathcal{T}(P^\bullet),\ \mathcal{F}(P^\bullet))$ in A-mod is split.

(2) $P^\bullet$ is called *splitting* if the induced torsion pair $(\mathcal{T}(Q^\bullet),\ \mathcal{F}(Q^\bullet))$ in B-mod is split.

Lemma 3.8. *Let* $P^\bullet$ *be a* 2*-term silting complex in* $\mathscr{K}^b(A\text{-proj})$.

(1) ([7, Lemma 5.5]) $P^\bullet$ *is splitting if and only if* $\mathrm{Ext}_A^2(\mathcal{T}(P^\bullet), \mathcal{F}(P^\bullet)) = 0$.

(2) ([7, Proposition 5.7]) *If $P^\bullet$ is separating, then $P^\bullet$ is a tilting complex.*

(3) ([21, Proposition 3.5]) *Suppose* $\mathrm{id}({}_AX) \leq 1$ *for each* $X \in \mathcal{F}(P^\bullet)$. *Then $P^\bullet$ is separating if and only if* $\mathrm{pd}({}_BY) \leq 1$ *for each* $Y \in \mathcal{T}(Q^\bullet)$.

Theorem 3.9. *Suppose A is an Artin algebra, $P^\bullet$ a 2-term silting complex, and* $B := \mathrm{End}_{\mathscr{D}^b(A)}(P^\bullet)$. *If $P^\bullet$ is separating and* $\mathrm{id}({}_AX) \leq 1$ *for each* $X \in \mathcal{F}(P^\bullet)$, *then* $\mathrm{ext.dim}(A) = \mathrm{ext.dim}(B)$.

Proof. It follows from Lemma 2.9 that the extension dimension and the weak resolution dimension of an algebra are the same. Thus we have to show that $\mathrm{w.resol.dim}(A) = \mathrm{w.resol.dim}(B)$.

It follows from $\mathrm{id}({}_AX) \leq 1$ for each $X \in \mathcal{F}(P^\bullet)$ that $\mathrm{Ext}^2_A(\mathcal{T}(P^\bullet), \mathcal{F}(P^\bullet)) = 0$. By Lemma 3.8(1), we get $P^\bullet$ is splitting, that is, the torsion pair $(\mathcal{T}(Q^\bullet), \mathcal{F}(Q^\bullet))$ in B-mod is split. By Definition 3.7(1), since $P^\bullet$ is separating, we know that the torsion pair $(\mathcal{T}(P^\bullet), \mathcal{F}(P^\bullet))$ in A-mod is split. Denote

$$\mathcal{H} := \mathrm{Hom}_{\mathscr{D}^b(A)}(P^\bullet, -) \quad \text{and} \quad \mathcal{E} := \mathrm{Hom}_{\mathscr{D}^b(A)}(P^\bullet, -[1]).$$

By Lemma 3.6(7), we have

$$\mathcal{H} : \mathcal{T}(P^\bullet) \xrightarrow{\simeq} \mathcal{F}(Q^\bullet) \text{ and } \mathcal{E} : \mathcal{F}(P^\bullet) \xrightarrow{\simeq} \mathcal{T}(Q^\bullet)$$

as additive categories.

We first prove that $\mathrm{w.resol.dim}(A) = 0$ if and only if $\mathrm{w.resol.dim}(B) = 0$. Suppose $\mathrm{w.resol.dim}(A) = 0$. By Lemmas 2.4(1) and 2.9, the weak resolution dimension of an algebra is equal to 0 if and only if it is representation-finite. Thus A is representation-finite, that is, the number of non-isomorphic indecomposable A-modules is finite. It follows from $\mathcal{H} : \mathcal{T}(P^\bullet) \xrightarrow{\simeq} \mathcal{F}(Q^\bullet)$ and $\mathcal{E} : \mathcal{F}(P^\bullet) \xrightarrow{\simeq} \mathcal{T}(Q^\bullet)$ as additive categories that the number of non-isomorphic indecomposable B-modules in $\mathcal{F}(Q^\bullet)$ is equal to the number of non-isomorphic indecomposable A-modules in $\mathcal{T}(P^\bullet)$; the number of non-isomorphic indecomposable B-modules in $\mathcal{T}(Q^\bullet)$ is equal to the number of non-isomorphic indecomposable A-modules in $\mathcal{F}(P^\bullet)$. Thus the number of non-isomorphic indecomposable B-modules in $\mathcal{F}(Q^\bullet)$ or $\mathcal{T}(Q^\bullet)$ is finite. Note that the torsion pair $(\mathcal{T}(Q^\bullet), \mathcal{F}(Q^\bullet))$ in B-mod is split, namely each indecomposable B-module lies either in $\mathcal{T}(Q^\bullet)$ or in $\mathcal{F}(Q^\bullet)$. Thus the number of non-isomorphic indecomposable B-modules is finite, that is, B is representation-finite. By Lemmas 2.4(1) and 2.9, $\mathrm{w.resol.dim}(B) = 0$. Similarly, if $\mathrm{w.resol.dim}(B) = 0$, then $\mathrm{w.resol.dim}(A) = 0$. Thus $\mathrm{w.resol.dim}(A) = 0$ if and only if $\mathrm{w.resol.dim}(B) = 0$.

Next, suppose $\mathrm{w.resol.dim}(A) \geq 1$ and $\mathrm{w.resol.dim}(B) \geq 1$. Set $m := \mathrm{w.resol.dim}(A)$. By Definition 2.6, there exists an A-module M such that $\mathrm{w.resol.dim}(A) = M\text{-}\mathrm{w.resol.dim}(A)$. Define $N := B \oplus \mathcal{H}(M)$. We shall show $N\text{-}\mathrm{w.resol.dim}(B) \leq m$.

By Lemma 2.7,

$$N\text{- w.resol.dim}(B) = \sup\{N\text{- w.resol.dim}({}_BY) \mid Y \in B\text{-ind}\},$$

where B-ind stands for the set of isomorphism classes of indecomposable finitely generated B-modules. Note that the torsion pair $(\mathcal{T}(Q^\bullet), \mathcal{F}(Q^\bullet))$ in B-mod is split, that is, each indecomposable B-modules lies either in $\mathcal{T}(Q^\bullet)$ or in $\mathcal{F}(Q^\bullet)$. Thus we have to show that $N\text{- w.resol.dim}({}_BY) \leq m$ for each $Y \in \mathcal{T}(Q^\bullet)$ or $\mathcal{F}(Q^\bullet)$.

Indeed, if $Y_0 \in \mathcal{F}(Q^\bullet)$, then it follows from $\mathcal{H} : \mathcal{T}(P^\bullet) \xrightarrow{\simeq} \mathcal{F}(Q^\bullet)$ that there exists $U \in \mathcal{T}(P^\bullet)$ such that $\mathcal{H}(U) \simeq Y_0$. Also M-w.resol.dim$(A) = m$. By Definition 2.6, there is an exact sequence

$$0 \longrightarrow M_m \longrightarrow M_{m-1} \longrightarrow \cdots \longrightarrow M_0 \longrightarrow U \oplus U' \longrightarrow 0$$

in A-mod for some A-module U' such that all $M_i \in \mathrm{add}({}_AM)$. Equivalently, there is an exact sequence

$$0 \to K_{i+1} \to M_i \xrightarrow{f_i} K_i \to 0 \tag{3.7}$$

in A-mod such that $M_i \in \mathrm{add}({}_AM)$ for each $0 \leq i \leq m-1$, where $K_0 := U \oplus U'$ and $K_m := M_m$. Since $P^\bullet$ is 2-term complex in $\mathscr{K}^b(A\text{-proj})$, we have, for any $W \in A$-mod, $\mathrm{Hom}_{\mathscr{D}^b(A)}(P^\bullet, W[j]) = 0$ for any $j \neq -1, 0$. For $0 \leq i \leq m-1$, applying the functor $\mathcal{H}$ to the sequence (3.7), we obtain the following exact sequence in B-mod:

$$0 \longrightarrow \mathcal{H}(K_{i+1}) \longrightarrow \mathcal{H}(M_i) \xrightarrow{\mathcal{H}(f_i)} \mathcal{H}(K_i) \longrightarrow \mathcal{E}(E_{i+1}) \longrightarrow \mathcal{E}(M_i) \longrightarrow \mathcal{E}(K_i) \longrightarrow 0 \tag{3.8}$$

Let C_i be the cokernel of $\mathcal{H}(f_i)$. By the long exact sequence (3.8), we have an exact sequence

$$0 \longrightarrow C_i \longrightarrow \mathcal{E}(E_{i+1}) \longrightarrow \mathcal{E}(M_i) \longrightarrow \mathcal{E}(K_i) \longrightarrow 0.$$

By Lemma 3.8(3), $\mathrm{pd}({}_BZ) \leq 1$ if $Z \in \mathcal{T}(Q^\bullet)$. Note that $\mathcal{E}(E_{i+1}), \mathcal{E}(M_i), \mathcal{E}(K_i)$ are in $\mathcal{T}(Q^\bullet)$, and so have projective dimension no more than 1. Thus $\mathrm{pd}({}_BC_i) \leq 1$. Let $0 \to Q_{i,1} \to Q_{i,0} \to C_i \to 0$ be a projective resolution of C_i with $Q_{i,j} \in B$-proj. Similar to the proof of the horseshoe lemma, we have the following commutative diagram

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{H}(K_{i+1}) & \longrightarrow & \mathcal{H}(K_{i+1}) \oplus Q_{i,1} & \longrightarrow & Q_{i,1} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{H}(M_i) & \longrightarrow & \mathcal{H}(M_i) \oplus Q_{i,0} & \longrightarrow & Q_{i,0} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Im}\mathcal{H}(f_i) & \longrightarrow & \mathcal{H}(K_i) & \longrightarrow & C_i & \longrightarrow & 0, \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

where $\mathrm{Im}\mathcal{H}(f_i)$ is the image of $\mathcal{H}(f_i)$. Then we have a short exact sequence

$$0 \to \mathcal{H}(K_{i+1}) \oplus Q_{i,1} \to \mathcal{H}(M_i) \oplus Q_{i,0} \to \mathcal{H}(K_i) \to 0. \tag{3.9}$$

Combine these short exact sequences (3.9) for $0 \leq i \leq m-1$, we get a long exact sequence

$$
\begin{aligned}
0 &\longrightarrow \mathcal{H}(M_m) \oplus Q_{m-1,1} \longrightarrow \cdots \longrightarrow \mathcal{H}(M_1) \oplus Q_{1,0} \oplus Q_{0,1} \longrightarrow \mathcal{H}(M_0) \oplus Q_{0,0} \\
&\longrightarrow \mathcal{H}(U) \oplus \mathcal{H}(U') \longrightarrow 0.
\end{aligned} \tag{3.10}
$$

Note $N = B \oplus \mathcal{H}(M)$. It follows from $M_i \in \mathrm{add}({}_AM)$ and $Q_{ij} \in \mathrm{add}({}_BB)$ that $\mathcal{H}(M_i), Q_{ij} \in \mathrm{add}({}_BN)$ for $0 \leq i \leq m-1, j = 0, 1$. By Definition 2.6 and the long exact sequence (3.10), we have

$$N\text{- w.resol.dim}({}_B\mathcal{H}(U)) \leq m.$$

It follows from $Y_0 \simeq \mathcal{H}(U)$ as B-modules that

$$N\text{- w.resol.dim}({}_BY_0) = N\text{- w.resol.dim}({}_B\mathcal{H}(U)) \leq m.$$

Thus we obtain

$$N\text{- w.resol.dim}({}_BY) \leq m, \ \forall\, Y \in \mathcal{F}(Q^\bullet).$$

On the other hand, by Lemma 3.8(3), we get $\mathrm{pd}({}_BY) \leq 1$ for each $Y \in \mathcal{T}(Q^\bullet)$. Also $B \in \mathrm{add}({}_BN)$. Then

$$N\text{- w.resol.dim}({}_BY) \leq \mathrm{pd}({}_BY) \leq 1 \leq m, \ \forall\, Y \in \mathcal{T}(Q^\bullet).$$

Thus we obtain $N\text{- w.resol.dim}({}_BY) \leq m$ for each $Y \in \mathcal{T}(Q^\bullet)$ or $\mathcal{F}(Q^\bullet)$. Then

$$N\text{- w.resol.dim}(B) \leq m.$$

By Definition 2.6, we have $\text{w.resol.dim}(B) \leq N\text{-w.resol.dim}(B)$. Then

$$\text{w.resol.dim}(B) \leq m = \text{w.resol.dim}(A).$$

Dually, we can prove $\text{w.resol.dim}(A) \leq \text{w.resol.dim}(B)$. Thus $\text{w.resol.dim}(A) = \text{w.resol.dim}(B)$. Finally, we have $\text{ext.dim}(A) = \text{ext.dim}(B)$ by Lemma 2.9. □

The following corollary is an immediate consequence of the above theorem.

Corollary 3.10. *Let A be an Artin algebra.*

(1) *Suppose that T is a separating and splitting tilting A-module with $\text{pd}({}_AT) \leq 1$. Then $\text{ext.dim}(A) = \text{ext.dim}(\text{End}_A(T))$.*

(2) *Suppose that $P^\bullet$ is a 2-term separating and splitting silting complex. Then $\text{ext.dim}(A/I) = \text{ext.dim}(\text{End}_A(H^0(P^\bullet)))$, where $I := \text{ann}_A(H^0(P^\bullet))$ is the annihilator of $H^0(P^\bullet)$.*

Proof. (1) Let $P^\bullet$ be the minimal projective resolution of T. Clearly, $\text{End}_{\mathscr{D}^b(A)}(P^\bullet) \simeq \text{End}_A(T)$ as algebras and $P^\bullet$ is a 2-term silting complex with

$$\mathcal{T}(P^\bullet) = \{U \in A\text{-mod} \mid \text{Ext}^1_A(T, U) = 0\} \text{ and } \mathcal{F}(P^\bullet) = \{V \in A\text{-mod} \mid \text{Hom}_A(T, V) = 0\}.$$

By [1, Theorem 3.6, p. 49], we have $\text{id}({}_AV) = 1$ for all $V \in \mathcal{F}(P^\bullet)$. Then

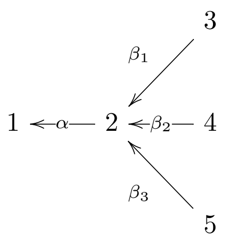
$$\text{ext.dim}(A) = \text{ext.dim}(\text{End}_{\mathscr{D}^b(A)}(P^\bullet)) = \text{ext.dim}(\text{End}_A(T))$$

follows from Theorem 3.9.

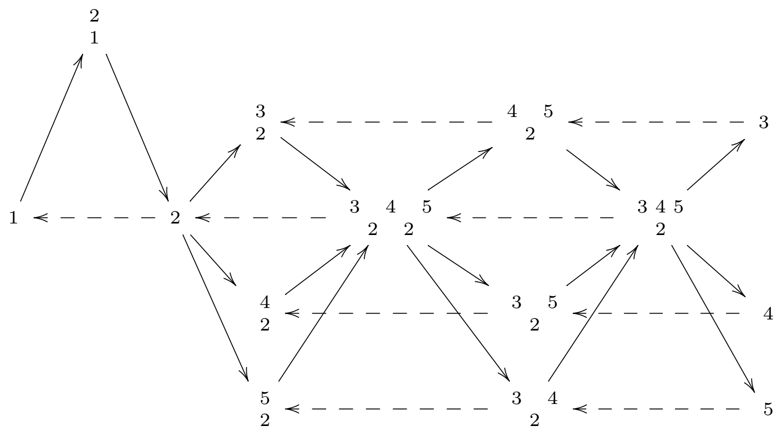
(2) By [21, Proposition 5.1], $H^0(P^\bullet)$ is a separating and splitting tilting A/I-module. Then the statement in (2) follows from (1). □

The following example illustrates that the assumptions in Theorem 3.9 are necessary.

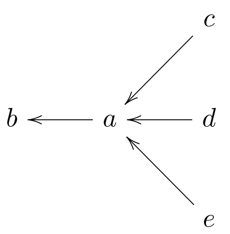
Example 3.11. Let A be an algebra over a field k given by the following quiver Q_A



with relations $\alpha\beta_i = 0,\ 1 \leq i \leq 3$. The Aulander-Reiten quiver of A-mod is as follows:



Thus A is representation-finite and $\text{ext.dim}(A) = 0$ by Lemma 2.4(1). Let $T := 2 \oplus \begin{smallmatrix} 2 \\ 1 \end{smallmatrix} \oplus \begin{smallmatrix} 3 \\ 2 \end{smallmatrix} \oplus \begin{smallmatrix} 4 \\ 2 \end{smallmatrix} \oplus \begin{smallmatrix} 5 \\ 2 \end{smallmatrix}$. By calculation, we obtain that T is a tilting module with $\text{pd}({}_AT) = 1$. Let $P^\bullet$ be the projective resolution of T. Then $P^\bullet$ is a 2-term tilting complex and $B := \text{End}_{\mathscr{D}^b(A)}(P^\bullet)$ is the path algebra given by the quiver Q_B



Since the underlying graph of Q_B is Euclidean, we know that B is a hereditary k-algebra of infinite representation type and $\text{ext.dim}(B) = 1$ by Lemmas 2.4(1) and 2.4(3).

By the definition of $Q^\bullet$ in Lemma 3.6, we can get $Q^\bullet$ is the 2-term tilting complex over B given by the direct sums of the following two complexes

$$0 \longrightarrow b \longrightarrow \begin{smallmatrix} a \\ b \end{smallmatrix} \longrightarrow 0, \qquad 0 \longrightarrow b \oplus \begin{smallmatrix} c \\ a \\ b \end{smallmatrix} \oplus \begin{smallmatrix} d \\ a \\ b \end{smallmatrix} \oplus \begin{smallmatrix} e \\ a \\ b \end{smallmatrix} \longrightarrow 0 \longrightarrow 0,$$

and $A \simeq \text{End}_{\mathscr{D}^b(B)}(Q^\bullet)$ as algebras. Since $\text{gl.dim}(B) = 1$, we have $\text{Ext}^2_B(\mathcal{T}(Q^\bullet), \mathcal{F}(Q^\bullet)) = 0$. By Lemma 3.8(1), $Q^\bullet$ is splitting and $P^\bullet$ is separating.

(1) For the 2-term silting complex $P^\bullet$, an easy calculation shows that $\mathcal{F}(P^\bullet) = \text{add}({}_AS(1))$ and $\text{id}({}_AS(1)) = 2$. Since $\text{gl.dim}(B) = 1$, we have $\text{pd}({}_BY) \le 1$ for each $Y \in \mathcal{T}(Q^\bullet)$. By Lemma 3.8(3), we know that $P^\bullet$ is separating. Due to $\text{ext.dim}(A) = 0 \neq 1 = \text{ext.dim}(B)$, this example shows that the homological dimension restriction on $\mathcal{F}(P^\bullet)$ cannot be removed.

(2) For the 2-term silting complex $Q^\bullet$, $(\mathcal{T}(Q^\bullet), \mathcal{F}(Q^\bullet))$ in B-mod is not split, namely it is not separating. Although $\mathrm{id}({}_BY) \leq 1$ for every $Y \in \mathcal{F}(Q^\bullet)$, $\mathrm{ext.dim}(A) = 0 \neq 1 = \mathrm{ext.dim}(B)$. This implies that the separability condition in Theorem 3.9 is necessary.

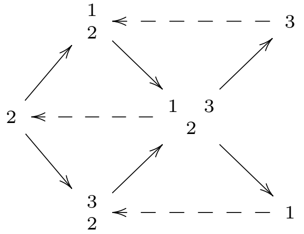
(3) Note that $P^\bullet$ is 2-term tilting complex and $|\,\mathrm{ext.dim}(A) - \mathrm{ext.dim}(B)| = 1$. Thus this example also illustrates that the inequality in Theorem 3.4 can be an equality.

To illustrate Theorem 3.9, we give the following example. In addition, this example also implies that the inequality in Theorem 3.4 could be strict.

Example 3.12. Let k be a field, and A be the path algebra given by the quiver

$$1 \xrightarrow{\alpha} 2 \xleftarrow{\beta} 3 \ .$$

The Auslander-Reiten quiver is given by



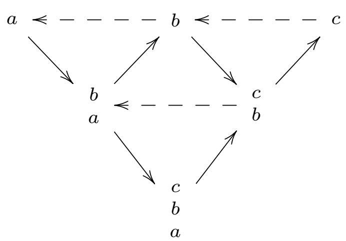
Let $P^\bullet$ be a complex given by the direct sums of the following two complexes

$$0 \longrightarrow 2 \longrightarrow \begin{matrix}1\\2\end{matrix} \longrightarrow 0, \quad 0 \longrightarrow 2 \oplus \begin{matrix}3\\2\end{matrix} \longrightarrow 0 \longrightarrow 0.$$

By calculation, $P^\bullet$ is a 2-term tilting complex over A, and $B := \mathrm{End}_{\mathscr{D}^b(A)}(P^\bullet)$ is a path algebra given by the quiver

$$a \longleftarrow b \longleftarrow c \ .$$

The Auslander-Reiten quiver is given by



By direct computation, $Q^\bullet$ (defined in Lemma 3.6) is the complex given by the direct sums of the following complexes

$$0 \longrightarrow a \longrightarrow \begin{smallmatrix} b \\ a \end{smallmatrix} \longrightarrow 0, \quad 0 \longrightarrow 0 \longrightarrow \begin{smallmatrix} b \\ a \end{smallmatrix} \oplus \begin{smallmatrix} c \\ b \\ a \end{smallmatrix} \longrightarrow 0.$$

Further, the induced torsion pair $(\mathcal{T}(P^\bullet), \mathcal{F}(P^\bullet))$ in A-mod is given by

$$\mathcal{T}(P^\bullet) = \mathrm{add}(\, {\scriptstyle 1} \,), \quad \mathcal{F}(P^\bullet) = \mathrm{add}(\, {\scriptstyle 2} \oplus \begin{smallmatrix} 1 \\ 2 \end{smallmatrix} \oplus \begin{smallmatrix} 3 \\ 2 \end{smallmatrix} \oplus \begin{smallmatrix} 1 & & 3 \\ & 2 & \end{smallmatrix} \oplus {\scriptstyle 3} \,).$$

The induced torsion pair $(\mathcal{T}(Q^\bullet), \mathcal{F}(Q^\bullet))$ in B-mod is given by

$$\mathcal{T}(Q^\bullet) = \mathrm{add}(\begin{smallmatrix} b \\ a \end{smallmatrix} \oplus b \oplus \begin{smallmatrix} c \\ b \\ a \end{smallmatrix} \oplus \begin{smallmatrix} c \\ b \end{smallmatrix} \oplus c), \quad \mathcal{F}(Q^\bullet) = \mathrm{add}(\, a \,).$$

Clearly, $(\mathcal{T}(P^\bullet), \mathcal{F}(P^\bullet))$ and $(\mathcal{T}(Q^\bullet), \mathcal{F}(Q^\bullet))$ are split. Then the homological conditions in Theorem 3.9 is satisfied. Thus $\mathrm{ext.dim}(A) = \mathrm{ext.dim}(B)$ and $|\,\mathrm{ext.dim}(A) - \mathrm{ext.dim}(B)| = 0 < 1$. This implies the inequality in Theorem 3.4 can be strict.

4. Stable equivalences

In this section, we shall prove that the extension dimension is invariant under stable equivalence. We first recall some basic results about the stable equivalence of Artin algebras, as detailed in reference [3,8,12].

Let A be an Artin algebra over a fixed commutative Artin ring k. Recall that an A-module X is called a *generator* if $A \in \mathrm{add}(X)$, *cogenerator* if $\mathrm{D}(A_A) \in \mathrm{add}(X)$, and *generator-cogenerator* if it is both a generator and cogenerator for A-mod. We denoted by A-$\underline{\mathrm{mod}}$ the stable module category of A modulo projective modules. The objects are the same as the objects of A-mod, and for two modules X, Y in A-$\underline{\mathrm{mod}}$, their homomorphism set is $\underline{\mathrm{Hom}}_A(X, Y) := \mathrm{Hom}_A(X, Y)/\mathscr{P}(X, Y)$, where $\mathscr{P}(X, Y)$ is the subgroup of $\mathrm{Hom}_A(X, Y)$ consisting of the homomorphisms factorizing through a projective A-module. This category is usually called the *stable module category* of A. Dually, We denoted by A-$\overline{\mathrm{mod}}$ the stable module category of A modulo injective modules. Let τ_A be the Auslander-Reiten translation DTr. Then $\tau_A : A\text{-}\underline{\mathrm{mod}} \to A\text{-}\overline{\mathrm{mod}}$ be an equivalence as additive categories (see [3, Chapler IV.1]). Two algebras A and B are said to be *stably equivalent* if the two stable categories A-$\underline{\mathrm{mod}}$ and B-$\underline{\mathrm{mod}}$ are equivalent as additive categories.

Next, suppose that $F : A\text{-}\underline{\mathrm{mod}} \to B\text{-}\underline{\mathrm{mod}}$ is an stable equivalence. Then the following functor

$$F' := \tau_B \circ F \circ \tau_A^{-1} : A\text{-}\overline{\mathrm{mod}} \longrightarrow B\text{-}\overline{\mathrm{mod}}$$

is equivalent as additive categories. Moreover, there are one-to-one correspondences

$$F : A\text{-mod}_{\mathscr{P}} \longrightarrow B\text{-mod}_{\mathscr{P}} \quad \text{and} \quad F' : A\text{-mod}_{\mathscr{I}} \longrightarrow B\text{-mod}_{\mathscr{I}},$$

where $A\text{-mod}_{\mathscr{P}}$ (respectively, $A\text{-mod}_{\mathscr{I}}$) stands for the full subcategory of A-mod consisting of modules without nonzero projective (respectively, injective) summands. We also use F (respectively, F') to denote the induce map A-mod $\to B$-mod which takes projectives (respectively, injectives) to zero.

Recall that a simple A-module S is called a *node* of A if it is neither projective nor injective, and the middle term of the almost split sequence starting at S is projective; a node S in A-mod is said to be an *F-exceptional node* if $F(S) \not\simeq F'(S)$. Let $\mathfrak{n}_F(A)$ be the set of isomorphism classes of F-exceptional nodes of A. Since $\mathfrak{n}_F(A)$ is a subset of all simple modules, $\mathfrak{n}_F(A)$ is a finite set. Note that X is indecomposable, non-projective, non-injective, and not a node, then $F(X) \simeq F'(X)$ (see [2, Lemma 3.4] or [3, Chapter X.1.7, p. 340]). Then $\mathfrak{n}_F(A)$ and the set of isomorphism classes of indecomposable, non-projective, non-injective A-modules U such that $F(U) \not\simeq F'(U)$, coincide.

Let $F^{-1} : B\text{-}\underline{\text{mod}} \to A\text{-}\underline{\text{mod}}$ be a quasi-inverse of F. Then we use $\mathfrak{n}_{F^{-1}}(B)$ to denote the the set of isomorphism classes of F^{-1}-exceptional nodes of B.

In the following, let

$$\triangle_A := \mathfrak{n}_F(A)\dot{\cup}(\mathscr{P}_A \setminus \mathscr{I}_A) \quad \text{and} \quad \triangledown_A := \mathfrak{n}_F(A)\dot{\cup}(\mathscr{I}_A \setminus \mathscr{P}_A),$$

where $\dot{\cup}$ stands for the disjoint union of sets; $\mathscr{P}_A$ and $\mathscr{I}_A$ stand for the set of isomorphism classes of indecomposable projective and injective A-modules, respectively. By $\triangle_A^c$ we mean the class of indecomposable, non-injective A-modules which do not belong to $\triangle_A$.

Remark 4.1. Each module $X \in A\text{-mod}_{\mathscr{I}}$ admits a unique decomposition (up to isomorphism)

$$X \simeq X^c \oplus X^{\triangle}$$

with $X^c \in \text{add}(\triangle_A^c)$ and $X^{\triangle} \in \text{add}(\triangle_A)$.

Let $\mathcal{GCN}_F(A)$ be the class of basic A-modules X which are generator-cogenerators with $\mathfrak{n}_F(A) \subseteq \text{add}(X)$, that is,

$$\mathcal{GCN}_F(A) = \{M' \oplus M_0 \in A\text{-mod} \mid M' \oplus M_0 \text{ is basic}\},$$

where M_0 is the unique (up to isomorphism) basic module with

$$\text{add}(M_0) = \text{add}(A) \cup \text{add}(\mathrm{D}(A_A)) \cup \text{add}(\mathfrak{n}_F(A)).$$

We say a module is *basic* if it is a direct sum of pairwise non-isomorphic indecomposable modules.

We define the following correspondences:

$$\Phi : A\text{-mod} \longrightarrow B\text{-mod}, \quad X \mapsto F(X) \oplus \bigoplus_{Q \in \mathscr{P}_B} Q,$$

$$\Psi : B\text{-mod} \longrightarrow A\text{-mod}, \quad Y \mapsto F^{-1}(Y) \oplus \bigoplus_{P \in \mathscr{P}_A} P.$$

Lemma 4.2. ([8, Lemma 4.10]) (1) *There exist one-to-one correspondences*

$$F : \bigtriangledown_A \longrightarrow \bigtriangledown_B, \quad F' : \triangle_A \longrightarrow \triangle_B \quad \text{and} \quad F' : \triangle_A^c \longrightarrow \triangle_B^c.$$

(2) *The correspondences* Φ *and* Ψ *restrict to one-to-one correspondences between* $\mathcal{GCN}_F(A)$ *and* $\mathcal{GCN}_{F^{-1}}(B)$. *Moreover, if* $X \in \mathcal{GCN}_F(A)$, *then* $\Phi(X) \simeq F'(X) \oplus \bigoplus_{J \in \mathscr{I}_B} J$.

Recall that an exact sequence $0 \to X \xrightarrow{f} Y \to Z \xrightarrow{g} 0$ in A-mod is called *minimal* ([30]) if it has no a split exact sequence as a direct summand, that is, there does not exist isomorphisms u, v, w such that the following diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \longrightarrow & 0 \\ & & \downarrow u & & \downarrow v & & \downarrow w & & \\ 0 & \longrightarrow & X_1 \oplus X_2 & \xrightarrow{\begin{pmatrix} f_1 & 0 \\ 0 & f_2 \end{pmatrix}} & Y_1 \oplus Y_2 & \xrightarrow{\begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix}} & Z_1 \oplus Z_2 & \longrightarrow & 0 \end{array}$$

is row exact and commute, where $Y_2 \neq 0$ and $0 \to X_2 \xrightarrow{f_2} Y_2 \xrightarrow{g_2} Z_2 \to 0$ is split. The next lemma shows that the stable functor has certain "exactness" property.

Lemma 4.3. *Let* Z *be an* A*-module without nonzero projective summands, and let*

$$0 \longrightarrow X \oplus X' \longrightarrow Y \oplus P \xrightarrow{g} Z \longrightarrow 0$$

be a minimal short exact sequence in A-mod *such that* $X \in \text{add}(\triangle_A^c)$, $X' \in \text{add}(\triangle_A)$, $Y \in A\text{-mod}_{\mathscr{P}}$ *and* $P \in \text{add}({}_AA)$. *Then there exists a minimal short exact sequence*

$$0 \longrightarrow F(X) \oplus F'(X') \longrightarrow F(Y) \oplus Q \xrightarrow{g'} F(Z) \longrightarrow 0$$

in B-mod *such that* $Q \in \text{add}({}_BB)$ *and* $g' = F(g)$ *in* B-$\underline{\text{mod}}$.

Proof. Note that if ${}_AZ$ is indecomposable and non-projective, then the statement in Lemma 4.3 has been proved in [8, Lemma 4.13], which is valid also for decomposable module having no nonzero projective summands by checking the argument there. □

Lemma 4.4. *Let $M \in \mathcal{GCN}_F(A)$. Then M-w.resol.dim$(A) = \Phi(M)$-w.resol.dim(B).*

Proof. We first claim that for each $X \in A$-mod,

$$\Phi(M)\text{- w.resol.dim}({}_BF(X)) \leq M\text{- w.resol.dim}({}_AX).$$

This can be proved by induction on M- w.resol.dim$({}_AX)$. In fact, if M- w.resol.dim$({}_AX) = 0$, then $X \in \text{add}({}_AM)$. So $F(X) \in \text{add}({}_B\Phi(M))$ by the definition of $\Phi(M)$ and $\Phi(M)$- w.resol.dim$({}_BF(X)) = 0$. Now suppose that for each ${}_AX$ with $0 \leq M$- w.resol.dim$({}_AX) \leq n-1$, we have $\Phi(M)$- w.resol.dim$({}_BF(X)) \leq M$- w.resol.dim$({}_AX)$. We shall show the conclusion for ${}_AX$ with M- w.resol.dim$({}_AX) = n$. By Definition 2.6, there exists an exact sequence

$$0 \longrightarrow M_n \xrightarrow{f_n} M_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_1} M_0 \xrightarrow{f_0} X \oplus X' \longrightarrow 0$$

in A-mod for some A-module X' such that $M_i \in \text{add}({}_AM)$ for $0 \leq i \leq n$. Let K is the kernel of f_0. Then M- w.resol.dim$({}_AK) \leq n-1$ and we have a short exact sequence

$$0 \longrightarrow K \longrightarrow M_0 \longrightarrow X \oplus X' \longrightarrow 0 \tag{4.1}$$

in A-mod. we can decompose the short exact sequence (4.1) as the direct sums of the following two short exact sequences

$$0 \longrightarrow K_1 \longrightarrow W_1 \longrightarrow U_1 \longrightarrow 0, \tag{4.2}$$

$$0 \longrightarrow K_2 \longrightarrow W_2 \longrightarrow U_2 \longrightarrow 0 \tag{4.3}$$

in A-mod, namely there are isomorphisms λ, μ, ν with the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \longrightarrow & M_0 & \longrightarrow & X \oplus X' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\lambda} & & \downarrow{\scriptstyle\mu} & & \downarrow{\scriptstyle\nu} & & \\ 0 & \longrightarrow & K_1 \oplus K_2 & \longrightarrow & W_1 \oplus W_2 & \longrightarrow & U_1 \oplus U_2 & \longrightarrow & 0, \end{array} \tag{4.4}$$

in A-mod such that (4.2) is minimal and (4.3) is split. Since the sequence (4.2) is minimal, we have $K_1 \in A\text{-mod}_{\mathscr{I}}$. By Remark 4.1, we can write $K_1 \simeq K_1^c \oplus K_1^{\triangle}$ with $K_1^c \in \text{add}(\triangle_A^c)$ and $K_1^{\triangle} \in \text{add}(\triangle_A)$. We also write $W_1 := W_1^{\mathscr{P}} \oplus P$ with $W_1^{\mathscr{P}} \in A\text{-mod}_{\mathscr{P}}$ and $P \in A$-proj. Then we can write the following short exact sequence

$$0 \longrightarrow K_1^c \oplus K_1^{\triangle} \longrightarrow W_1^{\mathscr{P}} \oplus P \longrightarrow U_1 \longrightarrow 0 \tag{4.5}$$

for the sequence (4.2). Since the sequence (4.2) is minimal, we know that the sequence (4.5) is minimal and $U_1 \in A\text{-mod}_{\mathscr{P}}$. By Lemma 4.3, we have the following minimal exact sequence

$$0 \longrightarrow F(K_1^c) \oplus F'(K_1^{\triangle}) \longrightarrow F(W_1^{\mathscr{P}}) \oplus Q \longrightarrow F(U_1) \longrightarrow 0 \tag{4.6}$$

in B-mod such that $Q \in B$-proj. Note that K_1^c is a direct summand of K. It follows from M- w.resol.dim$({}_AK) \le n-1$ that M- w.resol.dim$({}_AK_1^c) \le n-1$. By induction hypothesis, we see $\Phi(M)$- w.resol.dim$({}_BF(K_1^c)) \le M$- w.resol.dim$({}_AK) \le n-1$. By Definition 2.6, there exists an exact sequence

$$0 \longrightarrow N_{n-1} \overset{g_{n-1}}{\longrightarrow} N_{n-2} \longrightarrow \cdots \longrightarrow N_1 \overset{g_1}{\longrightarrow} N_0 \overset{g_0}{\longrightarrow} F(K_1^c) \oplus Y' \longrightarrow 0 \tag{4.7}$$

in B-mod for some B-module Y' such that $N_i \in \mathrm{add}({}_B\Phi(M))$ for $0 \le i \le n-1$. Let J be the injective envelope of ${}_BY'$. By the exact sequences (4.6) and (4.7), we get the following long exact sequence

$$\begin{aligned}0 \to N_{n-1} \to \cdots \to N_1 \to N_0 \oplus F'(K_1^{\triangle}) \to F(W_1^{\mathscr{P}}) \oplus Q \oplus J \oplus F(U_2)\\ \to F(U_1) \oplus \Omega^{-1}(Y') \oplus F(U_2) \to 0.\end{aligned} \tag{4.8}$$

By Lemma 4.2, it follows from $K_1^{\triangle} \in \mathrm{add}(\triangle_A)$ that $F'(K_1^{\triangle}) \in \mathrm{add}(\triangle_B)$ and $F'(K_1^{\triangle}) \in \mathrm{add}({}_B\Phi(M))$. Note that the short exact sequence (4.3) is split. Then U_2 is a direct summand of W_2. Also W_2 is a direct summand of M_0 and $M_0 \in \mathrm{add}({}_AM)$. Thus $U_2 \in \mathrm{add}({}_AM)$ and $F(U_2) \in \mathrm{add}({}_B\Phi(M))$. Note that $W_1^{\mathscr{P}}$ is a direct summand of W_1 and W_1 is a direct summand of M_0 and $M_0 \in \mathrm{add}({}_AM)$. Thus $W_1^{\mathscr{P}} \in \mathrm{add}({}_AM)$ and $F(W_1^{\mathscr{P}}) \in \mathrm{add}({}_B\Phi(M))$. It follows from the sequence (4.8) and $F(X) \in \mathrm{add}({}_BF(U_1 \oplus U_2))$ that $\Phi(M)$- w.resol.dim$({}_BF(X)) \le \Phi(M)$- w.resol.dim$({}_BF(U_1 \oplus U_2)) \le n$. Hence our claim is proved.

Next, We shall prove that $\Phi(M)$- w.resol.dim$(B) \le M$- w.resol.dim(A). Indeed, for $Y \in B$-mod, we can write $Y \simeq Y^{\mathscr{P}} \oplus Q'$ with $Y^{\mathscr{P}} \in B\text{-mod}_{\mathscr{P}}$ and $Q' \in B$-proj. Since $F : A\text{-mod}_{\mathscr{P}} \to B\text{-mod}_{\mathscr{P}}$ is an one-to-one correspondence, there exists $X \in A\text{-mod}_{\mathscr{P}}$ such that $F(X) \simeq Y^{\mathscr{P}}$ as B-modules. By the above claim, we get $\Phi(M)$- w.resol.dim$({}_BF(X)) \le M$- w.resol.dim$({}_AX) \le M$- w.resol.dim(A). Note that $Y \simeq F(X) \oplus Q'$ and $Q' \in \Phi(M)$. Then $\Phi(M)$- w.resol.dim$({}_BY) = \Phi(M)$- w.resol.dim$({}_BF(X)) \le M$- w.resol.dim(A). Thus $\Phi(M)$- w.resol.dim$(B) \le M$- w.resol.dim(A).

Similarly, for $N \in \mathcal{GCN}_{F^{-1}}(B)$, we can obtain that $\Psi(N)$- w.resol.dim$(A) \le N$- w.resol.dim(B). By Lemma 4.2(2), we can take $N_0 := \Phi(M)$. Then $M = \Psi(N_0)$ and

$$\begin{aligned}M\text{- w.resol.dim}(A) = \Psi(N_0)\text{- w.resol.dim}(A) &\le N_0\text{- w.resol.dim}(B)\\ &= \Phi(M)\text{- w.resol.dim}(B).\end{aligned}$$

Thus M- w.resol.dim$(A) = \Phi(M)$- w.resol.dim(B). □

Theorem 4.5. *Let A and B be stably equivalent Artin algebras. Then* ext.dim$(A) =$ ext.dim(B).

Proof. Let M_0 be the unique (up to isomorphism) basic module with

$$\mathrm{add}({}_AM_0) = \mathrm{add}(A) \cup \mathrm{add}(\mathrm{D}(A_A)) \cup \mathrm{add}(\mathfrak{n}_F(A)).$$

Then $\mathcal{GCN}_F(A) = \{M' \oplus M_0 \in A\text{-mod} \mid M' \oplus M_0 \text{ is basic}\}$. By Lemma 2.7(3), we have

$$\min\{M\text{- w.resol.dim}(A) \mid M \in A\text{-mod}\} = \min\{M\text{- w.resol.dim}(A) \mid M \in \mathcal{GCN}_F(A)\}.$$

Similarly, we have

$$\min\{N\text{- w.resol.dim}(B) \mid N \in B\text{-mod}\} = \min\{N\text{- w.resol.dim}(B) \mid N \in \mathcal{GCN}_{F^{-1}}(B)\}.$$

By Definition 2.6, we get the following equalities.

$$\begin{aligned}
&\mathrm{w.resol.dim}(A)\\
&\quad = \min\{M\text{- w.resol.dim}(A) \mid M \in A\text{-mod}\}\\
&\quad = \min\{M\text{- w.resol.dim}(A) \mid M \in \mathcal{GCN}_F(A)\}\\
&\quad = \min\{N\text{- w.resol.dim}(B) \mid N \in \mathcal{GCN}_{F^{-1}}(B)\} \quad \text{(by Lemmas 4.2(2) and 4.4)}\\
&\quad = \min\{N\text{- w.resol.dim}(B) \mid N \in B\text{-mod}\}\\
&\quad = \mathrm{w.resol.dim}(B).
\end{aligned}$$

By Lemma 2.9, we have $\mathrm{ext.dim}(A) = \mathrm{ext.dim}(B)$. □

Now, we deduce some consequences of Theorem 4.5.

Recall that a derived equivalence F between finite dimensional algebras A and B with a quasi-inverse G is called *almost ν-stable* [20] if the associated radical tilting complexes $T^\bullet$ over A and $\bar{T}^\bullet$ over B are of the form

$$\begin{aligned}
&T^\bullet : 0 \longrightarrow T^{-n} \longrightarrow \cdots T^{-1} \longrightarrow T^0 \longrightarrow 0 \ \text{ and}\\
&\bar{T}^\bullet : 0 \longrightarrow \bar{T}^0 \longrightarrow \bar{T}^1 \longrightarrow \cdots \longrightarrow \bar{T}^n \longrightarrow 0,
\end{aligned}$$

respectively, such that $\mathrm{add}(\bigoplus_{i=1}^n T^{-i}) = \mathrm{add}(\nu_A(\bigoplus_{i=1}^n T^{-i}))$ and $\mathrm{add}(\bigoplus_{i=1}^n \bar{T}^i) = \mathrm{add}(\nu_B(\bigoplus_{i=1}^n \bar{T}^i))$, where ν is the Nakayama functor. By [20, Theorem 1.1(2)], almost ν-stable derived equivalences induce special stable equivalences, namely stable equivalences of Morita type. Thus we have the following consequence of Theorem 4.5.

Corollary 4.6. *Let A and B be almost ν-stable derived equivalent finite dimensional algebras. Then* $\mathrm{ext.dim}(A) = \mathrm{ext.dim}(B)$.

Recall that given a finite dimensional algebra A over a filed k, $A \ltimes D(A)$, the trivial extension of A by $D(A)$ is the k-algebra whose underlying k-space is $A \oplus D(A)$, with multiplication given by

$$(a, f)(b, g) = (ab, fb + ag)$$

for $a, b \in A$, and $f, g \in D(A)$, where $D := \mathrm{Hom}_k(-, k)$. It is known that $A \ltimes D(A)$ is always symmetric, and therefore it is selfinjective.

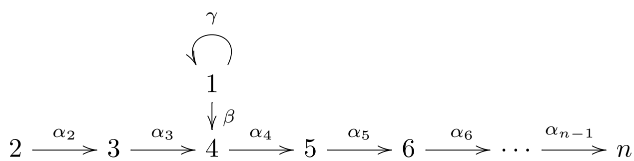
Corollary 4.7. *Let A and B be derived equivalent finite dimensional algebras. Then*

$$\mathrm{ext.dim}(A \ltimes D(A)) = \mathrm{ext.dim}(B \ltimes D(B)).$$

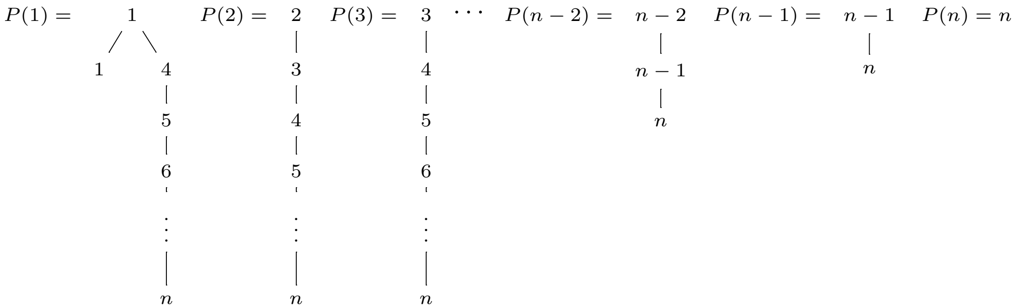
Proof. By a result of Rickard (see [34, Theorem 3.1]), which says that any derived equivalence between two algebras induces a derived equivalence between their trivial extension algebras, we obtain that $A \ltimes D(A)$ and $B \ltimes D(B)$ are derived equivalent. It follows from [20, Proposition 3.8] that every derived equivalence between two selfinjective algebras induces an almost ν-stable derived equivalence. Thus we have $\mathrm{ext.dim}(A \ltimes D(A)) = \mathrm{ext.dim}(B \ltimes D(B))$ by Corollary 4.6. □

In general, it is rather hard to compute the precise value of the extension dimension of a given algebra. However, we display an example to show how Theorem 4.5 can be applied to compute the extension dimensions of certain algebras. The example shows also that the method of computing extension dimensions by stable equivalences seems to be useful.

Example 4.8. Let k be a fixed field, $A = kQ_A/I$, where Q_A is the quiver



and I is generated by $\{\gamma^2, \beta\gamma\}$ with $n \geq 6$. The indecomposable projective left A-modules are



Then we have $\mathrm{pd}(S(1)) = \infty$, $\mathrm{pd}(S(i)) = 1$ for $2 \leq i \leq n-1$ and $\mathrm{pd}(S(n)) = 0$. Clearly, $\mathrm{gl.dim}(A) = \infty$. By calculation, we get that $\ell\ell(A) = n-1$ and $\ell\ell^{\infty}(A) = 2$, where $\ell\ell(A)$ and $\ell\ell^\infty(A)$ stand for the Loewy length and the infinite-layer length of A ([22,23]), respectively. By Lemma 2.4, we have

$$\text{ext.dim}(A) \le \min\{\text{gl.dim}(A), \ell\ell(A) - 1, \ell\ell^\infty(A) + 1\} = 3.$$

By the previous works in Lemma 2.4, one just get the upper bound for the extension dimension of A. Next, we shall compute the extension dimension of A by our results. By [29, Lemma 1], we know that $S(1)$ is a unique node of A. It follows from [29, Theorem 2.10] that A is stably equivalent to the path algebra B given by the following quiver Q_B:

$$\begin{array}{ccccccccccccc} & & & & 1' & & & & & & & & \\ & & & & \uparrow{\scriptstyle \delta} & & & & & & & & \\ & & & & 1 & & & & & & & & \\ & & & & \downarrow{\scriptstyle \beta} & & & & & & & & \\ 2 & \xrightarrow{\alpha_2} & 3 & \xrightarrow{\alpha_3} & 4 & \xrightarrow{\alpha_4} & 5 & \xrightarrow{\alpha_5} & 6 & \xrightarrow{\alpha_6} & \cdots & \xrightarrow{\alpha_{n-1}} & n. \end{array}$$

Since the underlying graph of Q_B is not Dynkin, B is representation-infinite and $\text{ext.dim}(B) = 1$ by Lemmas 2.4(1) and 2.4(3). By Theorem 4.5, we have $\text{ext.dim}(A) = \text{ext.dim}(B) = 1$.

Data availability

No data was used for the research described in the article.

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