

Prime numbers and the Riemann hypothesis

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ABSTRACT. Denote by ζ the Riemann zeta function. By considering the related prime zeta function, we demonstrate in this note that $\zeta(s) \neq 0$ for $\Re(s) > 1/2$, which proves the Riemann hypothesis.

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Introduction. Prime numbers have fascinated mathematicians since the ancient Greeks, and Euclid provided the first proof of their infinitude. Central to this subject is some innocent-looking infinite series known as the Riemann zeta function. This is a function of the complex variable s , defined in the half-plane $\Re(s) > 1$ by

$$\zeta(s) := \sum_{n=1}^{\infty} n^{-s}$$

and in the whole complex plane by analytic continuation. Euler noticed that the above series can be expressed as a product $\prod_p (1 - p^{-s})^{-1}$ over the entire set of primes, which entails that $\zeta(s) \neq 0$ for $\Re(s) > 1$. As shown by Riemann [2], $\zeta(s)$ extends to \mathbb{C} as a meromorphic function with only a simple pole at $s = 1$, with residue 1, and satisfies the functional equation $\xi(s) = \xi(1 - s)$, where $\xi(z) = \frac{1}{2}z(z-1)\pi^{-z/2}\Gamma(\frac{1}{2}z)\zeta(z)$ and $\Gamma(w) = \int_0^\infty e^{-x}x^{w-1}dx$. From the functional equation and the relationship between Γ and the sine function, it can be easily noticed that $\forall n \in \mathbb{N}$ one has $\zeta(-2n) = 0$, hence the negative even integers are referred to as the trivial zeros of ζ in the literature. The remaining zeros are all complex, and these are known as the nontrivial zeros. Riemann further

states, without sketching a proof, that in the range between 0 and τ the ξ function has about $\frac{\tau}{2\pi} \left(-1 + \log \frac{\tau}{2\pi} \right)$ nontrivial zeros. Define ρ to be a complex zero of ξ , hence a complex zero of ζ . The importance of the ρ 's in the distribution of primes can be clearly seen from the Riemann explicit formula

$$\psi(x) := \sum_{p^r \leq x} \log p = x - \frac{\zeta'(0)}{\zeta(0)} - \sum_{\rho} \frac{x^{\rho}}{\rho} + \frac{1}{2} \log(1 - x^{-2})$$

whenever x is half more than an integer and the summation on the left-hand side is over the prime powers p^r in the region specified, where $r \in \mathbb{N}$. In the literature (for example [6]), ψ is usually referred to as the Chebyshev ψ function after Russian mathematician P.L. Chebyshev who literally pioneered its study. It can be shown that

$$\psi(x) - x = O(x^{\Theta+\epsilon})$$

for every $\epsilon > 0$ if and only if $\zeta(s) \neq 0$ for $\Re(s) > \Theta$ [5, p.463], thus the bounds for $\psi(x) - x$ are controlled by the real parts of the ρ 's. Denote by $\pi(x)$ the number of primes not exceeding x . By partial summation, one finds that

$$\pi(x) = \frac{\psi(x)}{\log x} + \int_2^x \frac{\psi(t)}{t \log^2 t} dt + O(x^{1/2} \log x)$$

for $x \geq 2$, therefore the magnitude of $\pi(x)$ is also dependant on the real parts of the ρ 's, and the Prime Number Theorem that $\pi(x) \sim \text{Li}(x) = \lim_{\epsilon \rightarrow 0^+} \left(\int_0^{1-\epsilon} + \int_{1+\epsilon}^x \right) \frac{dt}{\log t}$ is equivalent to the nonvanishing of $\zeta(s)$ at $\Re(s) = 1$. This was first proved by Hadamard and de la Vallee Poussin working independently in 1896, (see for example [8, p.313]). However until now, there has never been found any absolute constant $\theta < 1$ such that $\zeta(s) \neq 0$ for $\Re(s) > \theta$. In particular, the Riemann hypothesis is equivalent to the above bound with $\Theta = \frac{1}{2}$. The interested reader can find far more thorough discussions of this problem in Titchmarsh [4] and/ or Borwein *et al* [7].

The main results of the paper and their proofs

Define $\pi(x)$ to be the number of primes not exceeding x , ζ to be the Riemann zeta function and $Li(x) = \lim_{\epsilon \rightarrow 0^+} \left(\int_0^{1-\epsilon} + \int_{1+\epsilon}^x \right) \frac{dt}{\log t}$ for $x > 1$. Throughout the following argument, σ shall denote the real part of the complex number s .

LEMMA 1. *For $\sigma > 1$, one has the identity*

$$s \int_1^\infty (\pi(x) - Li(x)) x^{-s-1} dx - \log((s-1)\zeta(s)) = \sum_{m=2}^\infty \frac{\mu(m)}{m} \log \zeta(ms), \quad (1)$$

where $\mu(n)$ denotes the Möbius function at n , which is equal to $(-1)^k$ if n is a square-free positive integer composed of k distinct prime factors and 0 otherwise.

PROOF. Let p be a prime. Consider the prime zeta function, defined by the infinite series

$$\sum_p \frac{1}{p^s} = \sum_{m=1}^\infty \frac{\mu(m)}{m} \log \zeta(ms) \quad (2)$$

for $\sigma > 1$ (see, for example [4, p.12]). Note that $\mu(1) = 1$ by convention. Applying partial summation to the left-hand side of (2) yields

$$s \int_1^\infty \pi(x) x^{-s-1} dx = \log \zeta(s) + \sum_{m=2}^\infty \frac{\mu(m)}{m} \log \zeta(ms). \quad (3)$$

for $\sigma > 1$. It is known [5, p.471] that $s \int_1^\infty Li(x) x^{-s-1} dx = -\log(s-1)$ for $\sigma > 1$, thus it follows from (3) that

$$s \int_1^\infty (\pi(x) - Li(x)) x^{-s-1} dx - \log((s-1)\zeta(s)) = \sum_{m=2}^\infty \frac{\mu(m)}{m} \log \zeta(ms) \quad (4)$$

for $\sigma > 1$, as desired. ■

LEMMA 2. *The domain of (4) extends by analytic continuation to the larger half-plane $\sigma > 1/2$.*

PROOF. Following Riemann [2], let $\pi_0(x)$ be equal to $\frac{1}{2}(\pi(x+0) + \pi(x-0))$ if x is a prime, and $\pi(x)$ otherwise. Define $\Pi(x) = \pi_0(x) + \frac{1}{2}\pi_0(x^{1/2}) + \frac{1}{3}\pi_0(x^{1/3}) + \dots$.

For $\sigma > 1$, we know [2, p.5] that $\log \zeta(s) = s \int_1^\infty \Pi(x) x^{-s-1} dx$ hence for $\sigma > 1$ it follows that

$$\log((s-1)\zeta(s)) = s \int_1^\infty (\Pi(x) - Li(x)) x^{-s-1} dx. \quad (5)$$

By combining (5) with Lemma 1, one finds that for $\sigma > 1$, the left-hand side of (4) is *identical* to $s \int_1^\infty (\pi(x) - \Pi(x)) x^{-s-1} dx$. Notice that $|\Pi(x) - \pi_0(x)| \ll x^{1/2}$ hence $|\pi(x) - \Pi(x)| \ll x^{1/2}$. We also have $|\mu(m) \log \zeta(ms)| \ll 2^{-m\sigma}$ as $m \rightarrow +\infty$ for $\sigma > 1/2$ [4, p.215]. These bounds imply that *both* sides of (4) are analytic and convergent functions of s whenever $\sigma > 1/2$, and the desired result follows. ■

COROLLARY 3. *Let*

$$F(s) = \int_1^\infty (\pi(x) - Li(x)) x^{-s-1} dx. \quad (6)$$

$F(s)$ converges on the *real* axis whenever $s > 1/2$.

PROOF. By Lemma 2, we know that identity (4) holds everywhere in the half-plane $\sigma > 1/2$. Recall from the proof of Lemma 2 that the right-hand side of (4) converges whenever $s > 1/2$. Thus the desired result follows immediately upon invoking the fact that $(s - 1)\zeta(s) > 0$ for every real $s > 0$ [5, Cor. 1.14]. ■

THEOREM 4. *One has $\zeta(s) \neq 0$ for $\sigma > 1/2$. That is, the Riemann hypothesis is true.*

PROOF. Define Θ to be the supremum of the real parts of the zeros of ζ , and suppose $\Theta > 1/2$. Note that the existence of such a Θ is guaranteed by [5, Cor. 1.10]. Theorem 15.2 of [5] tells us that $\pi(x) - Li(x) = \Omega_{\pm}(x^{\Theta-\epsilon})$ for every $\epsilon > 0$. Combining this with Theorem 1.3 of the same book, one finds that the abscissa of convergence of $F(s)$ is

$$\sigma_F := \limsup_{x \rightarrow \infty} \frac{\log |\pi(x) - Li(x)|}{\log x} \geq \Theta - \delta, \quad (7)$$

where δ is a fixed arbitrarily small positive number. By corollary 1.2 of [5], we know that $F(s_0)$ *does not* converge for any real number $s_0 < \sigma_F$. This entails the desired result that $\Theta \leq 1/2$, else inequality (7) would contradict Corollary 3. ■

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References

- [1] B.J. Conrey, *The Riemann Hypothesis*, Notices Amer. Math. Soc. **50**, (2003), 341 – 353.
- [2] B. Riemann, *Ueber die Anzahl Primzahlen unter einer gegebenen Grosse*, (1859).
- [3] E. Bombieri, *The Riemann Hypothesis - official problem description*, Clay Mathematics Institute, retrieved 2008-10-25. (Reprinted in Borwein *et al* [8, pp.94-105].)
- [4] E. C. Titchmarsh, *The theory of the Riemann zeta function*, second ed., The Clarendon Press Oxford University Press, New York, 1986, Edited and with a preface by D. R. Heath-Brown.
- [5] H. L. Montgomery, R. C. Vaughan, *Multiplicative number theory. I. Classical theory*. Cambridge Studies in Advanced Mathematics, **97**, Cambridge University Press, Cambridge, 2007.
- [6] J.B. Rosser and L. Schoenfeld, *Sharper bounds for the Chebyshev functions $\theta(x)$ and $\psi(x)$* , **II**, Math. Comp., **30**, (1976), 337 – 360.
- [7] P.B. Borwein, S. Choi, B. Rooney and A. Weirathmueller, *The Riemann Hypothesis: A resource for the aficionado and virtuoso alike*, (2008), Springer-Berlin.

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