Prime numbers and the Riemann hypothesis

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ABSTRACT. Denote by \( \zeta \) the Riemann zeta function. By considering the related prime zeta function, we demonstrate in this note that \( \zeta(s) \neq 0 \) for \( \Re(s) > 1/2 \), which proves the Riemann hypothesis.

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Introduction. Prime numbers have fascinated mathematicians since the ancient Greeks, and Euclid provided the first proof of their infinitude. Central to this subject is some innocent-looking infinite series known as the Riemann zeta function. This is is a function of the complex variable \( s \), defined in the half-plane \( \Re(s) > 1 \) by
\[
\zeta(s) := \sum_{n=1}^{\infty} n^{-s}
\]
and in the whole complex plane by analytic continuation. Euler noticed that the above series can be expressed as a product \( \prod_p (1 - p^{-s})^{-1} \) over the entire set of primes, which entails that \( \zeta(s) \neq 0 \) for \( \Re(s) > 1 \). As shown by Riemann [2], \( \zeta(s) \) extends to \( \mathbb{C} \) as a meromorphic function with only a simple pole at \( s = 1 \), with residue 1, and satisfies the functional equation \( \xi(s) = \xi(1 - s) \), where \( \xi(z) = \frac{1}{2} z(z - 1) \pi^{-\frac{1}{2} z} \Gamma\left(\frac{1}{2} z\right) \zeta(z) \) and \( \Gamma(w) = \int_{0}^{\infty} e^{-x} x^{w-1} \, dx \). From the functional equation and the relationship between \( \Gamma \) and the sine function, it can be easily noticed that \( \forall n \in \mathbb{N} \) one has \( \zeta(-2n) = 0 \), hence the negative even integers are referred to as the trivial zeros of \( \zeta \) in the literature. The remaining zeros are all complex, and these are known as the nontrivial zeros. Riemann further
states, without sketching a proof, that in the range between 0 and $\tau$ the $\xi$ function has about
\[ \frac{\tau}{2\pi} \left( -1 + \log \frac{\tau}{2\pi} \right) \]
nontrivial zeros. Define $\rho$ to be a complex zero of $\xi$, hence a complex zero of
$\zeta$. The importance of the $\rho$'s in the distribution of primes can be clearly seen from the Riemann
explicit formula
\[
\psi(x) := \sum_{p^j \leq x} \log p = x - \frac{\zeta'(0)}{\zeta(0)} - \sum_{\rho} \frac{x^\rho}{\rho} + \frac{1}{2} \log(1 - x^{-2})
\]
whenever $x$ is half more than an integer and the summation on the left-hand side is over the prime
powers $p^r$ in the region specified, where $j \in \mathbb{N}$. In the literature (for example [6]), $\psi$ is usually
referred to as the Chebyshev $\psi$ function after Russian mathematician P.L. Chebyshev who literally
pioneered its study. It can be shown that
\[
\psi(x) - x = O(x^{\Theta+\epsilon})
\]
for every $\epsilon > 0$ if and only if $\zeta(s) \neq 0$ for $\Re(s) > \Theta$ [5, p.463], thus the bounds for $\psi(x) - x$ are
controlled by the real parts of the $\rho$'s. Denote by $\pi(x)$ the number of primes not exceeding $x$. By
partial summation, one finds that
\[
\pi(x) = \frac{\psi(x)}{\log x} + \int_2^x \frac{\psi(t)}{t \log^2 t} \, dt + O(x^{1/2} \log x)
\]
for $x \geq 2$, therefore the magnitude of $\pi(x)$ is also dependant on the real parts of the $\rho$'s, and
the Prime Number Theorem that $\pi(x) \sim Li(x) = \lim_{\epsilon \to 0^+} \left( \int_0^{1-\epsilon} + \int_{1+\epsilon}^x \right) \frac{dt}{\log t}$ is equivalent to the
nonvanishing of $\zeta(s)$ at $\Re(s) = 1$. This was first proved by Hadamard and de la Vallee Poussin
working independently in 1896, (see for example [8, p.313]). However until now, there has never
been found any absolute constant $\theta < 1$ such that $\zeta(s) \neq 0$ for $\Re(s) > \theta$. In particular, the Riemann
hypothesis is equivalent to the above bound with $\Theta = \frac{1}{2}$. The interested reader can find far more
thorough discussions of this problem in Titchmarsh [4] and/ or Borwein et al [7].
The main results of the paper and their proofs

Define $\pi(x)$ to be the number of primes not exceeding $x$, $\zeta$ to be the Riemann zeta function and $\text{Li}(x) = \lim_{\epsilon \to 0^+} \left( \int_0^{1-\epsilon} + \int_{1+\epsilon}^x \right) \frac{dt}{\log t}$ for $x > 1$. Throughout the following argument, $\sigma$ shall denote the real part of the complex number $s$.

**LEMMA 1.** For $\sigma > 1$, one has the identity

$$s \int_1^\infty (\pi(x) - \text{Li}(x)) x^{-s-1} dx - \log((s-1)\zeta(s)) = \sum_{m=2}^\infty \frac{\mu(m)}{m} \log \zeta(ms),$$

(1)

where $\mu(n)$ denotes the Mobius function at $n$, which is equal to $(-1)^k$ if $n$ is a square-free positive integer composed of $k$ distinct prime factors and 0 otherwise.

**PROOF.** Let $p$ be a prime. Consider the prime zeta function, defined by the infinite series

$$\sum_p \frac{1}{p^s} = \sum_{m=1}^\infty \frac{\mu(m)}{m} \log \zeta(ms)$$

(2)

for $\sigma > 1$ (see, for example [4, p.12]). Note that $\mu(1) = 1$ by convention. Applying partial summation to the left-hand side of (2) yields

$$s \int_1^\infty \pi(x)x^{-s-1} dx = \log \zeta(s) + \sum_{m=2}^\infty \frac{\mu(m)}{m} \log \zeta(ms).$$

(3)

for $\sigma > 1$. It is known [5, p.471] that $s \int_1^\infty \text{Li}(x)x^{-s-1} dx = -\log(s-1)$ for $\sigma > 1$, thus it follows from (3) that

$$s \int_1^\infty (\pi(x) - \text{Li}(x)) x^{-s-1} dx - \log((s-1)\zeta(s)) = \sum_{m=2}^\infty \frac{\mu(m)}{m} \log \zeta(ms)$$

(4)

for $\sigma > 1$, as desired. ■

**LEMMA 2.** The domain of (4) extends by analytic continuation to the larger half-plane $\sigma > 1/2$.

**PROOF.** Following Riemann [2], let $\pi_0(x)$ be equal to $\frac{1}{2}(\pi(x+0)+\pi(x-0))$ if $x$ is a prime, and $\pi(x)$ otherwise. Define $\Pi(x) = \pi_0(x) + \frac{1}{2}\pi_0(x^{1/2}) + \frac{1}{3}\pi_0(x^{1/3}) + \cdots$.

For $\sigma > 1$, we know [2, p.5] that $\log \zeta(s) = s \int_1^\infty \Pi(x)x^{-s-1} dx$ hence for $\sigma > 1$ it follows that

$$\log((s-1)\zeta(s)) = s \int_1^\infty (\Pi(x) - \text{Li}(x)) x^{-s-1} dx.$$
By combining (5) with Lemma 1, one finds that for \( \sigma > 1 \), the left-hand side of (4) is identical to
\[
\int_1^\infty (\pi(x) - \Pi(x)) x^{-s-1} dx.
\]
Notice that \( |\Pi(x) - \pi_0(x)| \ll x^{1/2} \) hence \( |\pi(x) - \Pi(x)| \ll x^{1/2} \). We also have \( |\mu(m) \log(\zeta(ms))| \ll 2^{-m\sigma} \) as \( m \to +\infty \) for \( \sigma > 1/2 \) [4, p.215]. These bounds imply that both sides of (4) are analytic and convergent functions of \( s \) whenever \( \sigma > 1/2 \), and the desired result follows.

Let
\[
F(s) = s \int_1^\infty (\pi(x) - Li(x)) x^{-s-1} dx,
\]
so that (4) can be written as
\[
F(s) - \log((s-1) \zeta(s)) = \sum_{m=2}^{\infty} \frac{\mu(m)}{m} \log(\zeta(ms)).
\]

**COROLLARY 3.** \( F(s) \) converges on the real axis whenever \( s > 1/2 \).

**PROOF.** By Lemma 2, we know that identity (7) holds everywhere in the half-plane \( \sigma > 1/2 \).
Recall from the proof of Lemma 2 that the right-hand side of (7) converges whenever \( s > 1/2 \). Thus the desired result follows immediately from (7) upon invoking the fact that \( (s-1) \zeta(s) > 0 \) for every real \( s > 0 \) [5, Cor. 1.14].

**THEOREM 4.** One has \( \zeta(s) \neq 0 \) for \( \sigma > 1/2 \). That is, the Riemann hypothesis is true.

**PROOF.** Define \( \Theta \geq 1/2 \) to be the supremum of the real parts of the zeros of \( \zeta \). The existence of such a \( \Theta \) is guaranteed by [5, Cor. 1.10]. Since \( Li(x) = \sum_{2 \leq n \leq x} \frac{1}{\log n} + r(x) \) where \( r(x) = O(1) \), note that \( F(s) = F_0(s) + O(1) \) for \( \sigma > 0 \) where \( F_0(s) = s \int_1^\infty (\pi(x) - \sum_{2 \leq n \leq x} (\log n)^{-1}) x^{-s-1} dx \). It can be seen that \( F_0(s) \) is a Dirichlet integral from a Dirichlet series with coefficients \( a_n = \gamma(n) - (\log n)^{-1} \), where \( \gamma(n) = 1 \) if \( n \) is a prime and 0 otherwise. This allows us to apply Theorem 1.3 of [5] to \( F_0(s) \).
By combining it with Theorem 15.2 of the same book that \( \pi(x) - Li(x) = \Omega_{\pm}(x^{\Theta-\epsilon}) \) for every \( \epsilon > 0 \), we find that the abscissa of convergence of \( F_0(s) \) is
\[
\sigma_{F_0} := \limsup_{x \to \infty} \frac{\log |\pi(x) - Li(x)|}{\log x} \geq \Theta - \delta,
\]
where \( \delta \) is a fixed arbitrarily small positive number. By corollary 1.2 of [5], we know that \( F_0(\hat{s}) \) diverges for any \( \hat{s} \in (0, \sigma_{F_0}) \). Consequently, \( F(\hat{s}) \) would also diverge since \( F(\hat{s}) = F_0(\hat{s}) + O(1) \).
This entails the desired result that \( \Theta \leq 1/2 \), else inequality (8) would contradict Corollary 3.

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References


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