Prime numbers and the Riemann hypothesis

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ABSTRACT. Denote by ζ the Riemann zeta function. By considering the related prime zeta function, we demonstrate in this note that $\zeta(s) \neq 0$ for $\Re(s) > 1/2$, which proves the Riemann hypothesis.

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Introduction. Prime numbers have fascinated mathematicians since the ancient Greeks, and Euclid provided the first proof of their infinitude. Central to this subject is some innocent-looking infinite series known as the Riemann zeta function. This is is a function of the complex variable s, defined in the half-plane $\Re(s) > 1$ by

$$\zeta(s) := \sum_{n=1}^{\infty} n^{-s}$$

and in the whole complex plane by analytic continuation. Euler noticed that the above series can be expressed as a product $\prod_p (1 - p^{-s})^{-1}$ over the entire set of primes, which entails that $\zeta(s) \neq 0$ for $\Re(s) > 1$. As shown by Riemann [2], $\zeta(s)$ extends to $\mathbb{C}$ as a meromorphic function with only a simple pole at $s = 1$, with residue 1, and satisfies the functional equation $\xi(s) = \xi(1 - s)$, where $\xi(z) = \frac{1}{2}z(z-1)\pi^{-z/2}\Gamma(\frac{1}{2}z)\zeta(z)$ and $\Gamma(w) = \int_0^\infty e^{-x}x^{w-1}\mathrm{d}x$. From the functional equation and the relationship between Γ and the sine function, it can be easily noticed that $\forall n \in \mathbb{N}$ one has $\zeta(-2n) = 0$, hence the negative even integers are referred to as the trivial zeros of ζ in the literature. The remaining zeros are all complex, and these are known as the nontrivial zeros. Riemann further states, without sketching a proof, that in the range between 0 and τ the ξ function has about $\frac{\tau}{2\pi}\Big(-1+\log\frac{\tau}{2\pi}\Big)$ nontrivial zeros. Define ρ to be a complex zero of ξ, hence a complex zero of ζ. The importance of the $\rho's$ in the distribution of primes can be clearly seen from the Riemann explicit formula

$$\psi(x) := \sum_{p^j\le x}\log p = x - \frac{\zeta'(0)}{\zeta(0)} - \sum_{\rho}\frac{x^\rho}{\rho} + \frac{1}{2}\log(1-x^{-2})$$

whenever x is half more than an integer and the summation on the left-hand side is over the prime powers p^r in the region specified, where $j\in\mathbb{N}$. In the literature (for example [6]), ψ is usually referred to as the Chebyshev ψ function after Russian mathematician P.L. Chebyshev who literally pioneered its study. It can be shown that

$$\psi(x) - x = O(x^{\Theta+\epsilon})$$

for every $\epsilon > 0$ if and only if $\zeta(s)\neq 0$ for $\Re(s) > \Theta$ [5, p.463], thus the bounds for $\psi(x)-x$ are controlled by the real parts of the $\rho's$. Denote by $\pi(x)$ the number of primes not exceeding x. By partial summation, one finds that

$$\pi(x) = \frac{\psi(x)}{\log x} + \int_2^x \frac{\psi(t)}{t\log^2 t}\mathrm{d}t + O(x^{1/2}\log x)$$

for $x\ge 2$, therefore the magnitude of $\pi(x)$ is also dependant on the real parts of the $\rho's$, and the Prime Number Theorem that $\pi(x)\sim Li(x) = \lim_{\epsilon\to 0^+}\Big(\int_0^{1-\epsilon}+\int_{1+\epsilon}^x\Big)\frac{\mathrm{d}t}{\log t}$ is equivalent to the nonvanishing of $\zeta(s)$ at $\Re(s)=1$. This was first proved by Hadamard and de la Vallee Poussin working independently in 1896, (see for example [8, p.313]). However until now, there has never been found any absolute constant $\theta<1$ such that $\zeta(s)\neq 0$ for $\Re(s)>\theta$. In particular, the Riemann hypothesis is equivalent to the above bound with $\Theta=\frac{1}{2}$. The interested reader can find far more thorough discussions of this problem in Titchmarsh [4] and/ or Borwein *et al* [7].

The main results of the paper and their proofs

Define $\pi(x)$ to be the number of primes not exceeding x, ζ to be the Riemann zeta function and $Li(x) = \lim_{\epsilon \to 0^+} \left(\int_0^{1-\epsilon} + \int_{1+\epsilon}^{x} \right) \frac{\mathrm{d}t}{\log t}$ for $x > 1$. Throughout the following argument, σ shall denote the real part of the complex number s.

LEMMA 1. *For $\sigma > 1$, one has the identity*

$$s \int_1^\infty (\pi(x) - Li(x)) x^{-s-1} \mathrm{d}x - \log((s-1)\zeta(s)) = \sum_{m=2}^{\infty} \frac{\mu(m)}{m} \log \zeta(ms), \tag{1}$$

where $\mu(n)$ denotes the Mobius function at n, which is equal to $(-1)^k$ if n is a square-free positive integer composed of k distinct prime factors and 0 otherwise.

PROOF. Let p be a prime. Consider the prime zeta function, defined by the infinite series

$$\sum_p \frac{1}{p^s} = \sum_{m=1}^{\infty} \frac{\mu(m)}{m} \log \zeta(ms) \tag{2}$$

for $\sigma > 1$ (see, for example [4, p.12]). Note that $\mu(1) = 1$ by convention. Applying partial summation to the left-hand side of (2) yields

$$s \int_1^\infty \pi(x) x^{-s-1} \mathrm{d}x = \log \zeta(s) + \sum_{m=2}^{\infty} \frac{\mu(m)}{m} \log \zeta(ms). \tag{3}$$

for $\sigma > 1$. It is known [5, p.471] that $s \int_1^\infty Li(x) x^{-s-1} \mathrm{d}x = -\log(s-1)$ for $\sigma > 1$, thus it follows from (3) that

$$s \int_1^\infty (\pi(x) - Li(x)) x^{-s-1} \mathrm{d}x - \log((s-1)\zeta(s)) = \sum_{m=2}^{\infty} \frac{\mu(m)}{m} \log \zeta(ms) \tag{4}$$

for $\sigma > 1$, as desired. ■

LEMMA 2. *The domain of (4) extends by analytic continuation to the larger half-plane $\sigma > 1/2$.*

PROOF. Following Riemann [2], let $\pi_0(x)$ be equal to $\frac{1}{2}(\pi(x+0) + \pi(x-0))$ if x is a prime, and $\pi(x)$ otherwise. Define $\Pi(x) = \pi_0(x) + \frac{1}{2}\pi_0(x^{1/2}) + \frac{1}{3}\pi_0(x^{1/3}) + \cdots$.

For $\sigma > 1$, we know [2, p.5] that $\log \zeta(s) = s \int_1^\infty \Pi(x) x^{-s-1} \mathrm{d}x$ hence for $\sigma > 1$ it follows that

$$\log((s-1)\zeta(s)) = s \int_1^\infty (\Pi(x) - Li(x)) x^{-s-1} \mathrm{d}x. \tag{5}$$

By combining (5) with Lemma 1, one finds that for $\sigma > 1$, the left-hand side of (4) is *identical* to $s\int_1^\infty(\pi(x)-\Pi(x))x^{-s-1}\mathrm{d}x$. Notice that $|\Pi(x)-\pi_0(x)| \ll x^{1/2}$ hence $|\pi(x)-\Pi(x)| \ll x^{1/2}$. We also have $|\mu(m)\log\zeta(ms)| \ll 2^{-m\sigma}$ as $m \to +\infty$ for $\sigma > 1/2$ [4, p.215]. These bounds imply that *both* sides of (4) are analytic and convergent functions of s whenever $\sigma > 1/2$, and the desired result follows. ■

Let

$$F(s) = s\int_1^\infty (\pi(x) - Li(x))x^{-s-1}\mathrm{d}x, \tag{6}$$

so that (4) can be written as

$$F(s) - \log((s-1)\zeta(s)) = \sum_{m=2}^{\infty} \frac{\mu(m)}{m}\log\zeta(ms). \tag{7}$$

COROLLARY 3. *$F(s)$ converges on the real axis whenever $s > 1/2$.*

PROOF. By Lemma 2, we know that identity (7) holds everywhere in the half-plane $\sigma > 1/2$. Recall from the proof of Lemma 2 that the right-hand side of (7) converges whenever $s > 1/2$. Thus the desired result follows immediately from (7) upon invoking the fact that $(s-1)\zeta(s) > 0$ for every real $s > 0$ [5, Cor. 1.14]. ■

THEOREM 4. *One has $\zeta(s) \neq 0$ for $\sigma > 1/2$. That is, the Riemann hypothesis is true.*

PROOF. Define $\Theta \geq 1/2$ to be the supremum of the real parts of the zeros of ζ. The existance of such a Θ is guaranteed by [5, Cor. 1.10]. Since $Li(x) = \sum_{2\leq n\leq x}\frac{1}{\log n} + r(x)$ where $r(x) = O(1)$, note that $F(s) = F_0(s) + O(1)$ for $\sigma > 0$ where $F_0(s) = s\int_1^\infty(\pi(x) - \sum_{2\leq n\leq x}(\log n)^{-1})x^{-s-1}\mathrm{d}x$. It can be seen that $F_0(s)$ is a Dirichlet integral from a Dirichlet series with coefficients $a_n = \gamma(n) - (\log n)^{-1}$, where $\gamma(n) = 1$ if n is a prime and 0 otherwise. This allows us to apply Theorem 1.3 of [5] to $F_0(s)$. By combining it with Theorem 15.2 of the same book that $\pi(x) - Li(x) = \Omega_\pm(x^{\Theta-\epsilon})$ for every $\epsilon > 0$, we find that the abscissa of convergence of $F_0(s)$ is

$$\sigma_{F_0} := \limsup_{x\to\infty} \frac{\log|\pi(x) - Li(x)|}{\log x} \geq \Theta - \delta, \tag{8}$$

where δ is a fixed arbitrarily small positive number. By corollary 1.2 of [5], we know that $F_0(\hat{s})$ diverges for any $\hat{s} \in (0, \sigma_{F_0})$. Consequentially, $F(\hat{s})$ would also diverge since $F(\hat{s}) = F_0(\hat{s}) + O(1)$. This entails the desired result that $\Theta \leq 1/2$, else inequality (8) would contradict Corollary 3. ■

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